FINITELY ADDITIVE MEASURES IN THE ERGODIC THEORY OF MARKOV CHAINS. I †

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Abstract

We develop a new approach to the study of general Markov chains (MC), i.e. homogeneous Markov processes with discrete time on an arbitrary phase space. We extend the Markov operator from the traditional space of countably additive measures to the space of finitely additive measures. Given an arbitrary phase space, we construct its "gamma-compactification" to which we extend each Markov chain. We establish an isomorphism between the finitely additive Markov chains on the given space and the Feller chains on its "gamma-compactification." The study is carried out in the framework of the functional operator approach.
Key words and phrases: finitely additive measure, countably additive measure, Markov chain, Markov operators, arbitrary phase space, compactification of an arbitrary phase space, extension of a Markov chain to the compactification.

1. LANGUAGE, TOOLS, AND CONSTRUCTIONS

1. Finitely additive measures

Let X be an arbitrary set and let Σ be an algebra of its subsets. Denote by $\sigma(\Sigma)$ the σ-algebra generated by Σ, often assuming Σ itself to be a σ-algebra. If X is a topological space with topology $\tau = \tau_X$ then $\mathcal{A} = \mathcal{A}_X = \mathcal{A}_\tau$ and $\mathcal{B} = \mathcal{B}_X = \mathcal{B}_\tau$ are the Borel algebra and σ-algebra on X generated by τ. Throughout the article, we assume Σ to contain all singletons of X. We also assume that our topological space X is minimally T_1-separated, i.e., we suppose that all its singletons are closed. In this case, its Borel algebra $\mathcal{A}$ and σ-algebra $\mathcal{B}$ contain all singletons. All Hausdorff, regular, normal, and metric spaces are T_1-separated.

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Following the notations of [9], denote by $ba(X,\Sigma)$ the Banach space of all bounded finitely additive measures $\mu\colon \Sigma \to \mathbb{R}$ with norm the total variation of a measure on X $\big(\|\mu\| = \mathrm{Var}(\mu, X)\big)$ and by $ca(X,\Sigma)$, the Banach space of all bounded countably additive measures $\mu\colon \Sigma \to \mathbb{R}$ also with total variation as the norm. Finitely additive measures are also referred to as *charges* in the literature.

Definition 1.1 (see [29]). A nonnegative finitely additive measure $\mu \in ba(X,\Sigma)$ is called *purely finitely additive* if, for every countable additive measure $\lambda \in ca(X,\Sigma)$, $0 \le \lambda \le \mu$ implies $\lambda = 0$. A measure $\mu \in ba(X,\Sigma)$ is called purely finitely additive if both nonnegative measures μ^+ and μ^- of its Jordan decomposition $\mu = \mu^+ - \mu^-$ are purely finitely additive.

A special case of the following theorem was proven by A. D. Alexandrov in the first systematic study of finitely additive measures (see [1–3]). A general assertion was obtained by Yosida and Hewitt in [29].

Theorem 1.1. *Each finitely additive measure $\mu \in ba(X,\Sigma)$ is uniquely representable as $\mu = \mu_1 + \mu_2$, where $\mu_1 \in ca(X,\Sigma)$ is a countably additive measure and $\mu_2 \in ba(X,\Sigma)$ is a purely finitely additive measure.*

Note that the zero measure is countably and finitely additive. This is actually implied in Theorem 1.1 where, possibly, $\mu_2 = 0$.

Like countably additive measures, purely finitely additive measures form a vector subspace in $ba(X,\Sigma)$ which we denote by $pfa(X,\Sigma)$. Theorem 1.1 can be treated as an assertion on direct decomposition of the measure space:

$$ba(X,\Sigma) = ca(X,\Sigma) \oplus pfa(X,\Sigma).$$

A purely finitely additive measure vanishes on every finite set. The authors of [29] observe with a reference to Birkhoff that, on a countable set, there exist $2^{2^{\aleph_0}} = 2^c$ pairwise singular purely finitely additive measures (c is a continuum).

Consider the real line $X = \mathbb{R}$ with the Borel σ-algebra $\mathcal{B}$. Distinguish *two-valued* (1 or 0) purely finitely additive measures in $pfa(X,\mathcal{B})$. Among them, two types of measures are worth mentioning: the measures "concentrated" on a bounded segment $[a,b]$ and those "concentrated" arbitrarily far from the zero, i.e., "near infinity."

Two-valued measures $\mu \in pfa(X,\mathcal{B})$ of the first type. Every such measure μ has the only point $x_\mu \in \mathbb{R}$ (depending on μ) such that $\mu\big((x_\mu - \varepsilon, x_\mu + \varepsilon)\big) = \mu(\mathbb{R}) = 1$ for every $\varepsilon > 0$ but $\mu\big(\{x_\mu\}\big) = 0$. We can say that such μ "is concentrated" (or "fixes the full unit mass") arbitrarily close to the point x_μ but not at x_μ. Such a measure may be entirely "on the tail" of a sequence $x_n \to x_\mu$. Obviously, "near" a fixed point x, there are at least 2^c different two-valued purely finitely additive measures.

Two-valued measures $\mu \in pfa(X, \mathcal{B})$ *of the second type.* Each of these measures satisfies $\mu\big(\mathbb{R} \setminus [-n, n]\big) = \mu(\mathbb{R}) = 1$ for all $n \in \mathbb{N}$. They split into two classes: $\mu_1\big((-\infty, -n)\big) = 1$ and $\mu_2\big((n, \infty)\big) = 1$ for all $n \in \mathbb{N}$. The first measures are "concentrated near $-\infty$" and the second, "near $+\infty$." As for the first type, such measures can lie "on the tails" of sequences tending to $-\infty$ or $+\infty$ and there are at least 2^c such measures. A. D. Alexandrov (see [1–3]) calls them "unreal charges."

Purely finitely additive measures that are not two-valued can have a much more complicated structure. Note that a purely finitely additive measure has no support in the usual sense.

Given an arbitrary X, denote by $B(X)$ the Banach space of all bounded functions $f\colon X \to \mathbb{R}$ with the sup-norm

$$\|f\| = \sup_{x \in X} \big|f(x)\big|.$$

Suppose that we have an algebra Σ of subsets of X. Denote by $H(X, \Sigma)$ the vector space of all finite linear combinations of the characteristic functions χ_E of sets $E \in \Sigma$ and by $B(X, \Sigma)$, the closure of $H(X, \Sigma)$ in $B(X)$, i.e., we extend $H(X, \Sigma)$ by adjoining all the uniform limits of sequences in $H(X, \Sigma)$ thereto. Obviously, $B(X, \Sigma)$ is a Banach space and $H(X, \Sigma) \subset B(X, \Sigma) \subset B(X)$. If Σ is a σ-algebra then $B(X, \Sigma)$ is the Banach space of all Σ-measurable bounded functions. If Σ is not a σ-algebra then not all functions in $B(X, \Sigma)$ are Σ-measurable.

Recall that the integral with respect to a finitely additive measure, generalizing the Lebesgue integral, was constructed by Fichtenholz and Kantorovich and, simultaneously, by Hildebrandt in 1934. The most general theory for integration of unbounded functions with respect to unbounded finitely additive measures was suggested by Dunford and set forth systematically by Dunford and Schwartz in [9]. For us, it is enough to be able to integrate bounded functions with respect to bounded measures, which simplifies the problem because the construction of the usual Lebesgue integral is preserved.

Suppose that Σ is an algebra. Every $f \in B(X, \Sigma)$ is integrable with respect to each $\mu \in ba(X, \Sigma)$ for the algebra Σ (i.e., nonmeasurable functions are integrable too) and the σ-algebra Σ. The general properties of the integral $\int f d\mu$ are the same as those of the Lebesgue integral with respect to a countably additive measure. However, the case of a purely finitely additive μ has some specific features. For example, the analog to the Fubini Theorem for a double integral with respect to a purely finitely additive measure does not hold even if Σ is a σ-algebra. Denote the integral of a function f with respect to a measure μ on X by

$$\int_X f(x) d\mu(x) = \int f d\mu = \langle f, \mu \rangle = \langle \mu, f \rangle = f(\mu) = \mu(f).$$

In this article, we use the Banach space $C(X)$ of all bounded continuous functions on a topological space (X,τ). Clearly, $C(X)\subset B(X,\mathcal{A})\subset B(X,\mathcal{B})$. Moreover, every $f\in C(X)$ is integrable with respect to all $\mu\in ba(X,\mathcal{B})$ and $\mu\in ba(X,\mathcal{A})$.

Let X be an arbitrary topological space with topology τ and let Σ be an algebra of its subsets (not necessarily connected with τ). We now recall some conventional definitions and known facts and comment on them.

Definition 1.2. A set $E\in\Sigma$ is *regular* for a measure $\lambda\in ba(X,\Sigma)$ if, for every $\varepsilon>0$, there exist $F,G\in\Sigma$ such that $\overline{F}\subset E\subset \overset{\circ}{G}$ and $\operatorname{Var}(\mu, G\setminus F)<\varepsilon$ (here $\overline{F}$ is the closure of F and $\overset{\circ}{G}$ is the interior of G). Denote by $\mathcal{R}_\lambda$ the class of all regular sets for λ. A measure $\lambda\in ba(X,\Sigma)$ is called *regular* if each $E\in\Sigma$ is regular, i.e., $\mathcal{R}_\lambda=\Sigma$.

We use the standard notations: $rba(X,\Sigma)$ is the Banach space of all regular finitely additive bounded measures on (X,Σ) with norm the total variation of a measure; $rca(X,\Sigma)$ is the Banach space of all regular countably additive bounded measures on (X,Σ) with norm the total variation of a measure. We have the inclusions

$$rca(X,\Sigma)\subset rba(X,\Sigma)\subset ba(X,\Sigma);$$
$$rca(X,\Sigma)\subset ca(X,\Sigma)\subset ba(X,\Sigma).$$

It is known (see [9]) that, in a metric space X, if $\Sigma=\mathcal{B}$ then every countably additive measure is regular, i.e., $ca(X,\mathcal{B})=rca(X,\mathcal{B})$ for a metric X. However, some X admit nonregular countably additive measures, and some nonmetrizable X also satisfy $ca(X,\mathcal{B})=rca(X,\mathcal{B})$.

The finite additivity of μ does not imply its regularity even on $[0,1]$. There is a classical theorem by A. D. Alexandrov (see [1]) stating that if X is compact and a bounded finitely additive measure μ on $(X,\mathcal{B})$ is regular then μ is countably additive. Combining this with Hahn's extension theorem makes it possible to obtain a more general result which we will often use in the sequel.

Theorem 1.2 [9]. *Suppose that X is compact, Σ is an algebra, and $\mu\in rba(X,\Sigma)$. Then μ has a unique bounded regular countably additive extension to the σ-algebra $\sigma(\Sigma)$, i.e., we may assume that $rba(X,\Sigma)=rca\big(X,\sigma(\Sigma)\big)$. In particular, if X is compact then $rba(X,\mathcal{A})=rca(X,\mathcal{B})$.*

Thus, if a nonzero purely finitely additive measure can be "localized" on a compact space then this measure is not regular. In particular, a measure $\mu\geq 0$ on $(\mathbb{R},\mathcal{B})$ with $\mu\big((0,\varepsilon)\big)=1$ for all $\varepsilon>0$ is not regular.

If X is a topological space then, in many problems, the original measure $\lambda\in ba(X,\Sigma)$ can be replaced by a regular measure $\overline{\lambda}$ "stuck" to it in the topology τ_C generated by $C(X)$ in $ba(X,\Sigma)$. Such a procedure was studied in detail by the author in [33, 34]. In this connection, we now recall some facts that are necessary for the exposition.

Theorem 1.3 [33]. *Suppose that X is normal. Then, for every $\lambda \in ba(X,\mathcal{A})$, there exists a unique $\overline{\lambda} \in rba(X,\mathcal{A})$ such that $\int f d\lambda = \int f d\overline{\lambda}$ for every $f \in C(X)$. Moreover, $\lambda(X) = \overline{\lambda}(X)$; if $\lambda \geq 0$ then $\overline{\lambda} \geq 0$; if $\lambda \in ca(X,\mathcal{B})$ then $\overline{\lambda} \in rca(X,\mathcal{B})$ (as the extension of $\overline{\lambda} \in rca(X,\mathcal{A})$ to $\mathcal{B}$).*

Definition 1.3 [33]. Given $\lambda \in ba(X,\mathcal{A})$, we call the measure $\overline{\lambda} \in rba(X,\mathcal{A})$ corresponding to λ by Theorem 1.3 the *regularization* of λ.

Corollary 1.1. *If X is a Hausdorff compact space then, for every $\lambda \in ba(X,\mathcal{A})$, its regularization $\overline{\lambda}$ belongs to $rca(X,\mathcal{B})$.*

Definition 1.4 [33]. Assume that $\mu \in rba(X,\mathcal{A})$ and $\mu \geq 0$. The set $\mathcal{R}\{\mu\} = \big\{\lambda \in ba(X,\mathcal{A}) : \lambda \geq 0,\ \overline{\lambda} = \mu\big\}$ is called the *class of C-equivalent measures for μ.*

Theorem 1.4 [33]. *Let $\mu \in rba(X,\mathcal{A})$. The set $\mathcal{R}\{\mu\}$ is convex and compact in the τ_B-topology of $ba(X,\mathcal{B})$ (τ_B is the *-weak topology on $ba(X,\mathcal{B})$).*

It should be noted that, as a matter of fact, this natural pair of measures $(\lambda,\overline{\lambda})$ was used by many other authors as an intermediate technical tool (without studying the interrelation between λ and $\overline{\lambda}$ in detail).

Recall that there is a duality between the vector spaces of functions and measures (see [9]): $B^*(X,\Sigma) = ba(X,\Sigma)$ for an arbitrary (X,Σ) and $C^*(X) = rba(X,\mathcal{A})$ for a normal topological X, $C^*(X) = rca(X,\mathcal{B})$ for a Hausdorff compact space X, with equality signifying isometric isomorphisms and the spaces on the left-hand sides presenting the topological duals to the corresponding function spaces.

We will consider the four "natural" topologies on the vector spaces M of measures:

- τ_M is the strong (metric) topology on M;
- τ_{M^*} is the weak topology on M generated by the dual M^*;
- τ_B is the weak topology on M generated by $B(X,\Sigma)$ (i.e., the *-weak topology for $M = ba(X,\Sigma)$);
- τ_C is the weak topology on M generated by $C(X)$ for X a topological space.

These topologies are comparable for a fixed M: $\tau_C \prec \tau_B \prec \tau_{M^*} \prec \tau_M$. All topologies τ_{M^*}, τ_B, and τ_C are defined by a base for the Tychonoff topology with a neighborhood base of a point $\mu \in M$ of the form

$$V(\mu,\varepsilon;\xi_1\dots,\xi_n) = \Big\{\eta \in M : \big|\xi_i(\mu-\eta)\big| < \varepsilon,\ i = 1,2,\dots,n;\ n \in \mathbb{N},\ \varepsilon > 0\Big\}.$$

Here $\xi_1, \dots, \xi_n$ are linear functionals on the corresponding space M^*, $B(X, \Sigma)$, or $C(X)$. In the last two spaces, ξ_i is an arbitrary function $f \in B(X, \Sigma)$ or $f \in C(X)$ regarded as a linear functional on $ba(X, \Sigma)$ of the form $f(\mu) = \langle f, \mu \rangle = \int f d\mu$.

Below, we also use the notations $S_M = \{\mu \in M : \mu \geq 0, \mu(X) = 1\}$. Thus, S_{ca} is the set of all traditional (countably additive) probability measures on (X, Σ).

2. Banach limits of sequences of measures

The construction of a Banach limit was already used by A. D. Alexandrov in his study of set functions (see [2, 3]). This is quite natural because measures define linear functionals on the corresponding function spaces, while Banach limits were suggested by Banach as a rather general example of extension of linear functionals from a subspace to an entire space. Afterwards Banach limits were repeatedly applied to measure theory.

We observe that the idea of a Banach limit can be treated in different ways, which can be seen from the papers of various authors. For our purposes, we need to develop our own version that was published partly in [34]. Therefore, in what follows, we have to indicate some key points of this preprint (for the proofs the reader is referred to [34]).

Let ℓ_∞ be the space of all bounded numeric sequences with the sup-norm and let c be the space of all convergent sequences with the sup-norm. Clearly, $c \subset \ell_\infty$.

Definition 2.1. A linear functional $\xi \in c^*$ is called the *functional of the limit of a sequence* if the relation $\xi(\alpha) = \lim_{n\to\infty} \alpha_n$ holds for every $\alpha = (\alpha_1, \alpha_2, \dots) \in c$. We denote this functional on c by $\xi = \mathrm{Lim} \in c^*$, $\mathrm{Lim}(\alpha) = \mathrm{Lim}(\alpha_n) = \lim_{n\to\infty} \alpha_n$.

Every continuous extension of Lim from c to ℓ_∞ preserving the unit norm (generally speaking, such an extension is not unique) is called a *Banach limit functional* on ℓ_∞ and denoted by LIM; the class of all Banach limit functionals on ℓ_∞ is denoted by $\mathcal{L}$. For specific $\alpha = (\alpha_1, \alpha_2, \dots) \in \ell_\infty$ and $\mathrm{LIM} \in \mathcal{L}$, we call $\mathrm{LIM}(\alpha)$ the *Banach limit* (corresponding to the functional LIM) of α.

Under relevant conditions, we sometimes call a concrete functional LIM itself a Banach limit. Curiously, the functional ξ such that $\xi(\alpha) = \frac{1}{2}(\underline{\lim}\, \alpha_n + \overline{\lim}\, \alpha_n)$ for all $\alpha \in \ell_\infty$ is not a Banach limit.

Definition 2.2. Suppose that (X, Σ) is arbitrary and $\mu_n \in ba(X, \Sigma)$ is such that $\|\mu_n\| \leq M < \infty$, $n = 1, 2, \dots$. We call each measure $\mu \in ba(X, \Sigma)$ meeting the inequalities $\underline{\lim}\, \mu_n(E) \leq \mu(E) \leq \overline{\lim}\, \mu_n(E)$ for all $E \in \Sigma$ a *Banach* *limit of the sequence of measures* $\{\mu_n\}$ and denote it by $\text{LIM}(\mu_n)$. Denote by $L\{\mu_n\}$ the set of all Banach limits of a sequence $\{\mu_n\}$.

This definition is natural due to the fact that, as demonstrated in [34], all above-mentioned measures μ are obtained as Banach limits of the numeric sequences $\{\mu_n(E)\}$, $E \in \Sigma$.

Theorem 2.1. *Let* $\mu_n \in ba(X, \Sigma)$, $n = 1, 2, \ldots$. *For all* $E \in \Sigma$ *and* $r \in [\underline{\lim}\,\mu_n(E), \overline{\lim}\,\mu_n(E)]$, *there exists* $\mu \in L\{\mu_n\}$ *such that* $\mu(E) = r$.

If all μ_n's are countably additive and $\mu \in L\{\mu_n\}$ then μ need not be countably additive.

Definition 2.3. Measures $\mu_1, \mu_2 \in ba(X, \Sigma)$ are said to be *singular* if there exists a set $E \in \Sigma$ with $\text{Var}(\mu_1, E) = 0$ and $\text{Var}(\mu_2, X \setminus E) = 0$.

A set $H \subset ba(X, \Sigma)$ is called a *set of pairwise singular measures* or just a *singular set* if every pair of measures in H are singular. A set $H \subset ba(X, \Sigma)$ is called a *set of jointly singular measures* if, for every measure $\mu \in H$, there exists a set $E \in \Sigma$ such that $\text{Var}(\mu, E) = 0$ and $\text{Var}(\eta, X \setminus E) = 0$ for all $\eta \in H \setminus \{\mu\}$.

If H is countable and consists of countably additive measures then pairwise singularity is equivalent to joint singularity. This fails in general.

Theorem 2.2. *Suppose that some measures* $\mu_n \in ba(X, \Sigma)$, $\|\mu_n\| \leq M < \infty$, $n = 1, 2, \ldots$, *are jointly singular. Then each measure* $\mu \in L\{\mu_n\}$ *is purely finitely additive.*

Theorem 2.2, together with the previous remarks, implies the following

Theorem 2.2′. *Assume that* $\mu_n \in ca(X, \Sigma)$ *with* $\|\mu_n\| \leq M < \infty$, $n = 1, 2, \ldots$, *are pairwise singular. Then each measure* $\mu \in L\{\mu_n\}$ *is purely finitely additive.*

Theorem 2.3. *Let* $\mu_n \in ba(X, \Sigma)$, $n = 1, 2, \ldots$, *and* $\mu_n \to \mu$ *in any of the topologies* τ_{ba}, τ_{ba^*}, *or* τ_B. *Then* $\mu \in L\{\mu_n\} = \{\mu\}$.

Theorem 2.4. *Let* X *be a normal topological space,* $\mu_n \in rba(X, \mathcal{A})$, $\mu_n \geq 0$, $n = 1, 2, \ldots$, *and* $\mu_n \to \mu \in rba(X, \mathcal{A})$ *in the* τ_C*-topology. Then* $L\{\mu_n\} \subset \mathcal{R}\{\mu\}$, *where* $\mathcal{R}\{\mu\}$ *is the class of* C*-equivalent measures for* μ.

Theorem 2.5. *Under the conditions of Theorem* 2.4, *we have* $\mu_n \to \lambda$ *in the* τ_C*-topology for each* $\lambda \in L\{\mu_n\}$; *moreover, there exists a base* β *of the topology of* X *such that* $\mu_n(E) \to \lambda(E) = \mu(E)$ *for all* $E \in \beta$.

Corollary 2.1. *Under the conditions of Theorem* 2.4, *the class of measures* $L\{\mu_n\}$ *can contain only one countably additive measure, namely,* μ.

Corollary 2.2. *Assume that* X *and* Σ *are infinite and an arbitrary sequence* $\{\mu_n\}$ *of measures has no* $\lim \mu_n(E)$ *for some* $E \in \Sigma$. *Then* $L\{\mu_n\}$ *has cardinality at least* 2^c.

Theorem 2.6. *Assume that* (X, Σ) *is arbitrary and* $\mu_n \in ba(X, \Sigma)$, $\|\mu_n\| \le M < \infty$, $n = 1, 2, \ldots$. *The set of all Banach limits* $L\{\mu_n\}$ *is convex and compact in the* τ_B*-topology.*

Let $\{\mu_n\}$ be a uniformly bounded sequence of measures in $ba(X, \Sigma)$. Denote by $\mathfrak{M}\{\mu_n\}$ the set of all limit points of $\{\mu_n\}$ in the τ_B-topology.

Theorem 2.7. *Assume that* (X, Σ) *is arbitrary and* $\mu_n \in ba(X, \Sigma)$, $\|\mu_n\| \le M < \infty$, $n = 1, 2, \ldots$. *Then* $\mathfrak{M}\{\mu_n\} \subset L\{\mu_n\}$, *i.e., all* τ_B*-limit measures of* μ_n *are Banach limits.*

Let X be a normal topological space and let $\{\mu_n\}$ be a uniformly bounded sequence of measures in $rba(X, \mathcal{A})$. Denote by $\mathfrak{N}\{\mu_n\}$ the set of all limit points of $\{\mu_n\}$ in the τ_C-topology.

Theorem 2.8. *Suppose that* X *is a normal space and* $\mu_n \in rba(X, \mathcal{A})$, $\|\mu_n\| \le M < \infty$, $n = 1, 2, \ldots$. *Then* $L\{\mu_n\} \subset \mathfrak{N}\{\mu_n\}$, *i.e., all the Banach limits of* $\{\mu_n\}$ *are its* τ_C*-limit measures.*

Generally speaking,

$$\mathfrak{M}\{\mu_n\} \ne L\{\mu_n\} \ne \mathfrak{N}\{\mu_n\}.$$

If a measure μ is τ_B- or τ_C-limit for $\{\mu_n\}$ then a subsequence μ_{n_i} τ_B- or τ_C-converging to μ does not necessarily exist.

Theorem 2.9. *Assume that* $\mu_n \in rba(X, \mathcal{A})$, $\mu_n \ge 0$, $n = 1, 2, \ldots$, *and a measure* μ *is* τ_C*-limit for* $\{\mu_n\}$ *in* $rba(X, \mathcal{A})$, *i.e.,* $\mu \in \mathfrak{N}\{\mu_n\}$. *Then*

$$\mu(F) \ge \underline{\lim}\, \mu_n(F) \quad \text{for all } F = \overline{F};$$

$$\mu(G) \le \overline{\lim}\, \mu_n(G) \quad \text{for all } G = \mathring{G}.$$

2. EXTENSION OF MARKOV OPERATORS TO THE SPACE OF FINITELY ADDITIVE MEASURES

3. Dual pairs of Markov operators

Let X be an arbitrary set and let Σ be a σ-algebra of its subsets which contains all singletons.

Definition 3.1. A *transition function* (*transition probability*) $p(x, E)$ on a measure space (X, Σ) is a mapping $p\colon X \times \Sigma \to [0, 1]$ satisfying the conventional conditions

$$p(\cdot, E) \in B(X, \Sigma), \quad E \in \Sigma;$$

$$p(x, \cdot) \in ca(X, \Sigma), \quad x \in X;$$

$$p(x, X) = 1, \quad x \in X.$$

A transition function defines a homogeneous Markov chain with discrete time (MC) on (X, Σ). Sometimes, we will use more exact terms and call such a transition function countably additive and the corresponding Markov chain, a countably additive MC.

Definition 3.2. By the *Markov* operators, we mean the two operators T and A defined explicitly as follows:

$$T\colon B(X,\Sigma) \to B(X,\Sigma), \quad (Tf)(x) = Tf(x) \stackrel{\text{def}}{=} \int_X f(y)p(x,dy),$$

where $f \in B(X,\Sigma)$, $x \in X$;

$$A\colon ca(X,\Sigma) \to ca(X,\Sigma), \quad (A\mu)(E) = A\mu(E) \stackrel{\text{def}}{=} \int_X p(x,E)\mu(dx),$$

where $\mu \in ca(X,\Sigma)$, $E \in \Sigma$.

These operators have been studied quite well. They are bounded linear operators with $\|T\| = \|A\| = 1$. Regarding the space of measures as a space of linear functionals in a function space, we can state the duality of T and A in a wide sense. Since $B^*(X,\Sigma) \neq ca(X,\Sigma)$ for infinite X, the operator A is not adjoint to T in the strict sense.

The operator T is positive, i.e., it maps the cone K^B of nonnegative functions $B(X,\Sigma)$ into itself. The cone K^B is solid, i.e., it has nonempty interior but, generally speaking, T does not map the interior of K^B into itself. Possibly, there exist $f \in K^B$ with $\|f\| = 1$ such that $Tf = 0$, i.e., T is not an isometry on K^B. Each Markov operator T has an interior fixed point in K^B: $f(x) \equiv 1 > 0$ for all $x \in X$, $f \in K^B$, $\|f\| = 1$, and $Tf = f$.

The operator A is also positive, i.e., it maps the cone K^{ca} of nonnegative measures in $ca(X,\Sigma)$ into itself. The cone K^{ca} is not solid, i.e., it has empty interior; and A is an isometry on K^{ca}, i.e., if $\mu \geq 0$ then $\|A\mu\| = \|\mu\| = \mu(X)$. Recall that

$$S_{ca} = \bigl\{\mu \in ca(X,\Sigma) : \mu \geq 0,\ \mu(X) = 1\bigr\} = \bigl\{\mu \in K^{ca} : \mu(X) = 1\bigr\}$$

is the set of all probability measures. Thus, $AS_{ca} \subset S_{ca}$. Note that A may fail to have a fixed point in K^{ca}, i.e. in S_{ca}. If there exists a fixed point $\mu = A\mu \in S_{ca}$ then such a measure is called an *invariant measure* of the operator A or a *stationary distribution* of the corresponding MC.

Assume that $\mu_0 \in S_{ca}$ and $\mu_n = A^n\mu_0 = A\mu_{n-1}$, $n = 1, 2, \ldots$. An MC can be identified with the sequence of probability measures $\{\mu_n\} = \bigl\{\mu_n(\mu_0)\bigr\}$ depending on the initial measure μ_0 as a parameter. Therefore, *every* MC *can be regarded as an iterative process generated by a positive linear operator on a space of measures.* This is the interpretation of an MC we keep to in this article.

The dual of $B(X,\Sigma)$ is the space of finitely additive measures $ba(X,\Sigma)$. Consequently, the adjoint T^* must be defined on $ba(X,\Sigma)$. If we want to apply the functional methods properly, we should clearly consider the problem of extending A to $ba(X,\Sigma)$ and thus complete the dual construction of the operators T and A. These arguments are already well known but even now the articles on MC's do not give a wide application of the full construction of T and T^*.

The first extensions of the Markov operator A to the space of finitely additive measures appeared in the articles by Foguel [11] (1962), [12] (1966) and Šidak [24] (1962). These articles give an example of occasional use of finitely additive measures as an auxiliary intermediate object. After that the idea that finitely additive measures are important in probability theory in general and theory of Markov processes in particular paved a road in these areas rather slowly. Though going back to the 70's, it was not until the 90's that the term "finitely additive probability" itself took up an appropriate place in scientific periodicals.

Considering the main problems of Markov chains theory, the author systematically uses the Markov operator A extended to the space of finitely additive measures, i.e., the pair T and T^*. These studies were started in the end of the 70's; the first results were published by the author in [30, 31] in 1981.

Since the following assertion is a folklore, we omit its easy proof.

Theorem 3.1. *For every countably additive* MC, *the Markov operator A of Definition* 3.2 *is uniquely extendable from $ca(X,\Sigma)$ to a linear operator $\widetilde{A}$ on $ba(X,\Sigma)$, preserving positivity, isometry on the cone, boundedness, the norm, and explicit form*

$$\widetilde{A}\colon ba(X,\Sigma) \to ba(X,\Sigma),$$

$$(\widetilde{A}\mu)(E) \stackrel{\text{def}}{=} \int_X p(x,E)\mu(dx), \quad \mu \in ba(X,\Sigma),\ E \in \Sigma.$$

Moreover, $\widetilde{A}$ is topologically adjoint to the operator T of Definition 3.2, *i.e.,* $T^* = \widetilde{A}$ *with* $B^*(X,\Sigma) = ba(X,\Sigma)$.

Definition 3.3. We call the extension $\widetilde{A}$ of the Markov operator A of Theorem 3.1 the (*finitely additive*) *extension of* A. Like A, we call $\widetilde{A}$ a *Markov operator.*

Below, we often identify $\widetilde{A}$ and A without specifying their domains of definition.

Suppose that $\mu_0 \in ba(X,\Sigma)$ is such that $\mu_0 \geq 0$ and $\|\mu_0\| = \mu_0(X) = 1$, i.e., $\mu_0 \in S_{ba}$. Then $\widetilde{A}$ generates the sequence of finitely additive measures $\mu_n = \widetilde{A}\mu_{n-1} = \widetilde{A}^n\mu_0 \in ba(X,\Sigma)$, $n = 1, 2, \ldots$. Following our ideology, *such an iterative process can be treated as a countably additive* MC *extended to the space of finitely additive measures.*

Emphasize that *we carry out a finitely additive extension of A and* MC *itself for a transition probability, which is still countably additive, i.e., we do not fall outside the limits of the conventional definition of* MC.

We will consider "Markov chains (processes)" with finitely additive transition probability in a separate section of this article.

In defining an MC, we have assumed that Σ is a σ-algebra of subsets in X, which corresponds to a strong tradition. However, this restriction is not important. Let Σ be an algebra of subsets in X. Then, by Hahn's extension theorem, each countably additive measure $\mu \in ca(X, \Sigma)$ is uniquely extendable to a countably additive measure $\widetilde{\mu}$ on $\sigma(\Sigma)$. Therefore, we may assume that $ca(X, \Sigma) = ca(X, \sigma(\Sigma))$. With natural clarifications, the Markov chain and its operators can also be defined in this case.

We now turn to the case of a topological phase space. Let X be topological space in which every singleton is closed (a T_1-separated space). The Borel algebra $\mathcal{A}$ and σ-algebra $\mathcal{B}$ contain all singletons. Consequently, we can also apply all the above arguments for an MC defined on $(X, \mathcal{B})$ or $(X, \mathcal{A})$. In the general theory, several principal types of MC's on $(X, \mathcal{B})$ are distinguished with some extra properties connected with endowing X with a topology. The most important of these types are the Feller MC's.

Definition 3.4. An MC defined on $(X, \mathcal{B})$ is called *Feller* if $TC(X) \subset C(X)$. The Markov operators corresponding to a Feller MC are also called *Feller.*

A Feller operator T can be regarded at the same time as $T\colon B(X, \mathcal{B}) \to B(X, \mathcal{B})$ and $T\colon C(X) \to C(X)$. Recall that we have $C^*(X) = rca(X, \mathcal{A})$ for every normal topological space X. Hence, the initial operator A before and after its extension to $ba(X, \mathcal{B})$ is not adjoint to the operator T on $C(X)$.

We confine exposition to normal topological spaces on which every countably additive measure is regular, i.e.,

$$ca(X, \mathcal{B}) = rca(X, \mathcal{B}).$$

This is always the case for metric spaces. Then if $\mu \in rba(X, \mathcal{A})$ and μ is countable additive then μ is extendable to $\mathcal{B}$ and $\mu \in rca(X, \mathcal{B})$.

The following assertion analogous to Theorem 3.1 is also well known and easy to prove.

Theorem 3.2. *For every countably additive Feller* MC, *the Markov operator A of Definition* 3.2 *is uniquely extendable from $ca(X, \mathcal{B}) = rca(X, \mathcal{B})$ to a linear operator $\widetilde{\widetilde{A}}$ on $rba(X, \mathcal{A})$ preserving positivity, isometry on the cone,* *boundedness, the norm, and explicit form*

$$\widetilde{\widetilde{A}}\colon rba(X,\mathcal{A}) \to rba(X,\mathcal{A}),$$

$$(\widetilde{\widetilde{A}}\mu)(E) = \widetilde{\widetilde{A}}\mu(E) \stackrel{\text{def}}{=} \int_X p(x,E)\mu(dx), \quad \mu \in rba(X,\mathcal{A}),\ E \in \mathcal{A}.$$

Moreover, $\widetilde{\widetilde{A}}$ *is topologically adjoint to* $T\colon C(X) \to C(X)$, *i.e.,* $T^* = \widetilde{\widetilde{A}}$. *In addition, the operator* $\widetilde{A}$ *of Theorem* 3.1 *and Definition* 3.3 *is an extension of* $\widetilde{\widetilde{A}}$ *from* $rba(X,\mathcal{A})$ *to* $ba(X,\mathcal{B})$.

Definition 3.5. We call $\widetilde{\widetilde{A}}$ the *regular finitely additive extension* of a Feller operator A and, sometimes, identify it with A.

Now, consider the case when X is a Hausdorff compact space. Such a space is normal and $ca(X,\mathcal{B}) = rca(X,\mathcal{B}) = rba(X,\mathcal{A})$. This implies the following obvious assertion.

Corollary 3.1. *Given a Feller* MC *on a Hausdorff compact space* $(X,\mathcal{B})$, *the Markov operator* $A\colon rca(X,\mathcal{B}) \to rca(X,\mathcal{B})$ *of Definition* 3.2 *coincides with its regular finitely additive extension* $\widetilde{\widetilde{A}}$ *of Definition* 3.5. *Moreover,* $A = \widetilde{\widetilde{A}}$ *is adjoint to* $T\colon C(X) \to C(X)$.

We now touch upon homogeneous Markov processes with continuous time generated by a transition function $p(t,x,E)$, where $x \in X$, $E \in \Sigma$, and $t > 0$. They also generate two semigroups of Markov operators $T^t\colon B(X,\Sigma) \to B(X,\Sigma)$ and $A^t\colon ca(X,\Sigma) \to ca(X,\Sigma)$ of the same analytic expression for every $t > 0$. Obviously, we can consider the finitely additive extension $\widetilde{A}^t$ to $ba(X,\Sigma)$ in the same manner as earlier for discrete time. Similarly, we consider the Feller process, in particular, on a compact space. In this article, we do not go into a special study of Markov processes with continuous time since their theory essentially differs from that of Markov chains with discrete time. However, some of the results that we obtain for chains below can be carried over to processes almost mechanically, which we will note in the relevant places.

4. Invariant measures of finitely additive extensions of Markov operators. Basic theorems

In Section 3, we passed from the traditional just "dual" pair of Markov operators to a pair of topologically adjoint Markov operators. This allows us to perfectly use the results of the Kreĭn–Rutman theory (see [18]) on the properties of positive linear operators. Since the operator $T\colon B(X,\Sigma) \to B(X,\Sigma)$ has an interior fixed point $f(x) \equiv 1$ in K^B, the Kreĭn–Rutman theorem [18, Theorem 3.1] implies directly (without extra arguments) that the adjoint operator $\widetilde{A}$ has a fixed point in K^{ba}. This is a key fact to us.

Theorem 4.1 (Basic Theorem I). *For every* MC *on an arbitrary measure space* (X, Σ), *there exists an invariant finitely additive measure*

$$\lambda \in ba(X, \Sigma), \quad \lambda \geq 0, \quad \lambda(X) = 1, \quad \lambda = \widetilde{A}\lambda,$$

i.e.,

$$\lambda \in S_{ba}, \quad \lambda(E) = \int p(x, E)\lambda(dx), \quad E \in \Sigma.$$

This theorem was proven by Šidak (see [24]), who was the first to consider the extension of the Markov operator to the space of finitely additive measures. However, the proof in [24] is rather complicated and does not involve positivity of the Markov operators. The fact that Theorem 4.1 is an easy corollary to the Kreĭn–Rutman theorem was noticed by the author in [31].

The following assertion was also obtained in [24].

Theorem 4.2. *Suppose that we have* $\lambda = \widetilde{A}\lambda$ *for an arbitrary* MC *and some* $\lambda \in S_{ba}$. *If* $\lambda = \lambda_1 + \lambda_2$ *is the decomposition of* λ *into the sum of a countably additive and purely finitely additive measures then* $\lambda_1 = \widetilde{A}\lambda_1$ *and* $\lambda_2 = \widetilde{A}\lambda_2$.

Thus, in many cases it suffices to consider only countably additive and purely finitely additive invariant measures separately.

Now, take a Feller MC. Applying the Kreĭn–Rutman theorem to the operators $T\colon C(X) \to C(X)$ and $\widetilde{\widetilde{A}}\colon rba(X, \mathcal{A}) \to rba(X, \mathcal{A})$, as a corollary we have the following important assertion.

Theorem 4.3 (Basic Theorem II). *For every Feller* MC *defined on a normal topological space* (X, τ), *there exists an invariant regular finitely additive measure*

$$\lambda \in rba(X, \mathcal{A}), \quad \lambda \geq 0, \quad \lambda(X) = 1, \quad \lambda = \widetilde{\widetilde{A}}\lambda,$$

i.e.,

$$\lambda \in S_{rba}, \quad \lambda(E) = \int p(x, E)\lambda(dx), \quad E \in \mathcal{A}.$$

Explicitly, there is no such result either in [24] or in Foguel's articles [11–13, 15]. However, it can be obtained by slightly modifying Foguel's results and he actually takes it into account. The fact that Theorem 4.3 is an easy corollary to the Kreĭn–Rutman theorem was observed by the author also in [31].

In [12], Foguel obtained an analogous assertion to Theorem 4.2.

Theorem 4.4. *Assume that a normal X satisfies the equality $ca(X,\mathcal{B}) = rca(X,\mathcal{B})$ and a Feller* MC *is given on $(X,\mathcal{B})$. If $\lambda = \tilde{\tilde{A}}\lambda \in S_{rba}$ and $\lambda = \lambda_1 + \lambda_2$ is the decomposition of λ into a countably additive measure λ_1 and a finitely additive measure λ_2 then $\lambda_1 = \tilde{\tilde{A}}\lambda_1$ and $\lambda_2 = \tilde{\tilde{A}}\lambda_2$.*

As in the case of the above Basic Theorems, the following assertion is immediate from the Kreĭn–Rutman theorem.

Theorem 4.5 (Basic Theorem III)**.** *For every Feller* MC *on a Hausdorff compact space $(X,\mathcal{B})$, there exists an invariant regular countably additive measure*

$$\lambda \in rca(X,\mathcal{B}), \quad \lambda \geq 0, \quad \lambda(X) = 1, \quad \lambda = A\lambda,$$

i.e.,

$$\lambda \in S_{rca}, \quad \lambda(E) = \int p(x,E)\lambda(dx), \quad E \in \mathcal{B}.$$

This assertion follows directly from the Basic Theorem II. For a compact metric space, Theorem 4.5 was first proven by Bebutov in 1948 (see [5]).

For each space of measures we use in this article, denote the set of positive normalized invariant measures for the Markov operator A by $\Delta_M = \{\mu \in S_M : \mu = A\mu\}$. For Δ_{ba}, we sometimes omit the index: $\Delta = \Delta_{ba}$. In particular, $\Delta_{pfa} = \{\mu \in \Delta : \mu \text{ is purely finitely additive}\}$. Let M_1 and M_2 be two measure spaces. Introduce the notation

$$\Delta_{M_1} \oplus \Delta_{M_2} = \{\mu = \alpha_1\mu_1 + \alpha_2\mu_2 : \mu_1 \in \Delta_{M_1}, \mu_2 \in \Delta_{M_2}; \\ 0 \leq \alpha_1, \alpha_2 \leq 1,\ \alpha_1 + \alpha_2 = 1\};$$

if $\Delta_{M_1} = \varnothing$ or $\Delta_{M_2} = \varnothing$ then we put $\alpha_i \equiv 0$ and $\alpha_i\mu_i \equiv 0$ for the corresponding i; if $\Delta_{M_1} = \Delta_{M_2} = \varnothing$ then $\Delta_{M_1} \oplus \Delta_{M_2} = \varnothing$. With some liberty, we can say that $\Delta_{M_1} \oplus \Delta_{M_2}$ are normalizations of elements of the direct sum of the sets Δ_{M_1} and Δ_{M_2}. We can now rewrite the above theorems in a brief form more convenient for the sequel as follows.

Theorem 4.1. *For every* MC, $\Delta = \Delta_{ba} \neq \varnothing$.

Theorem 4.2. *For every* MC, $\Delta = \Delta_{ba} = \Delta_{ca} \oplus \Delta_{pfa}$.

Theorem 4.3. *For every Feller* MC, $\Delta_{rba} \neq \varnothing$.

Theorem 4.4. *For every Feller* MC, $\Delta_{rba} = \Delta_{rca} \oplus \Delta_{pfa}$.

Theorem 4.5. *For every Feller* MC *on a Hausdorff compact space*, $\Delta_{rca} \neq \varnothing$.

We now turn to homogeneous Markov processes with continuous time. The Kreĭn–Rutman theorem was formulated in [18] for semigroups of adjoint operators. By analogy with the Basic Theorems I–III, as a corollary, we immediately have the following assertion, which we formulate omitting details.

Theorem 4.6 (Basic Theorem IV). *For every Markov process with continuous time, there exists an invariant finitely additive measure* $\lambda \in S_{ba}$, $\lambda = A^t \lambda$, *independent of* $t > 0$.

For a Feller process with continuous time on a normal space, there exists an invariant regular finitely additive measure $\lambda \in S_{rba}$, $\lambda = A^t \lambda$, *independent of* $t > 0$.

For a Feller Markov process with continuous time on a Hausdorff compact space, there exists an invariant regular countably additive measure $\lambda \in S_{rca}$, $\lambda = A^t \lambda$, *independent of* $t > 0$.

Theorem 4.6 as a corollary to the Kreĭn–Rutman theorem was pointed out in the author's preprint [34]. Note that, in the above-cited articles by other authors, there is neither an analog nor a special case of this assertion.

5. Finitely additive Markov chains

In the previous two sections, we have considered Markov chains with transition probability $p(x, E)$ countable with respect to the second argument. Although we extended Markov operators (i.e., Markov chains) to spaces of finitely additive measures, the transition probabilities and Markov chains themselves remained countably additive. By the intrinsic logic of development of mathematics, it would be natural to consider also finitely additive transition probabilities for MC's. At the same time, a real need arose to study finitely additive MC's.

In 1965, the famous monograph by Dubins and Savage [10] appeared, in which, apparently, finitely additive measures first acquired a "probabilistic" sense and the very title "probability." In [10], finitely additive measures were considered on a countable product of discrete spaces, playing the role of strategies in the problems of game theory. The monograph [10] gave rise immediately to a series of articles on the topic, mainly by students and followers of Dubins and Savage (see, for example, [8, 21]).

Apart from stochasticity, games also presuppose time that leads to using the theory of stochastic processes. Therefore, it was quite natural that Ramakrishnan's article [22] appeared which, in the framework of development of the ideas by Dubins and Savage, was the first to comprise the term "finitely additive Markov chains" already in the title. In [22], rather specific Markov chains with finitely additive transition function were studied in the strategy language. The phase space there is discrete and, in some assertions, countable. Later, some more articles were published in the context of Ramakrishnan's approach [22] including [23]. We observe that the methods of [22] and relevant articles are rather specific and cannot be carried over mechanically to more general Markov chains that have nothing to do with game theory.

Now, we extend the operator approach of the previous two sections to finitely additive MC's. It is worth noting that even defining MC has some problems to overcome.

Let X be an arbitrary set and let Σ be an algebra of its subsets which contains all singletons.

Definition 5.1. A *finitely additive transition function* (*transition probability*) $p(x, E)$ on a measure space (X, Σ) is a mapping $p\colon X \times \Sigma \to [0, 1]$ satisfying the conditions

$$\begin{aligned} &p(\cdot, E) \in B(X, \Sigma), \quad && E \in \Sigma; \\ &p(x, \cdot) \in ba(X, \Sigma), \quad && x \in X; \\ &p(x, X) = 1, && x \in X. \end{aligned}$$

Theorem 5.1. *A finitely additive transition function $p(x, E)$ defines the two integral operators*

$$(Tf)(x) = Tf(x) = \int f(y)p(x, dy), \qquad f \in B(X, \Sigma), \;\; x \in X,$$

$$(A\mu)(E) = A\mu(E) = \int p(x, E)\mu(dx), \quad \mu \in ba(X, \Sigma), \;\; E \in \Sigma.$$

Moreover,

$$T\colon B(X, \Sigma) \to B(X, \Sigma), \quad A\colon ba(X, \Sigma) \to ba(X, \Sigma),$$

T and A are linear, positive, and bounded, $\|T\| = \|A\| = 1$, and A is adjoint to T, i.e., $T^* = A$.

The proof is standard and similar to that of Theorem 3.1; therefore, we omit it.

Definition 5.2. Assume that (X, Σ) is endowed with a finitely additive transition probability. We call the operators T and A that correspond to it by Theorem 5.1 *finitely additive Markov operators.*

Suppose that a finitely additive transition function $p(x, E)$ is not countably additive. Then there exists a point $x_0 \in X$ such that $p(x_0, \cdot) \notin ca(X, \Sigma)$. Consider the Dirac measure $\mu = \delta_{x_0} \in ca(X, \Sigma)$. We have

$$A\mu = \int p(x, \cdot)\, \delta_{x_0}(dx) = p(x_0, \cdot) \notin ca(X, \Sigma).$$

Thus, generally speaking, the Markov operator A of a finitely additive transition function does not map countably additive measures to countably additive measures, i.e., $ca(X, \Sigma)$ is not invariant under A. The rest of the easy properties of finitely additive Markov operators are the same as the main properties of countably additive operators. Therefore, we preserve the terminology.

In Section 3, in considering a countably additive MC, we gave its functional interpretation rather than a rigorous definition. We had in mind that there are exhaustive definitions of all notions concerning MC's in the language of random variables (elements) which are in a correspondence with countably additive measures.

In the countably additive case, we have a fundamentally new situation. *No "random variables" that would recall habitual objects can be assigned to purely finitely additive measures.* At any rate, there is no such theory, and no direct analog is possible (except for some very special cases). Consequently, we need to give not an "interpretation" but a definition of such MC's "from scratch." For us, the most logical way is to take the "functional treatment" of Section 3 as such a definition. We were not able to give this definition before Theorem 5.1.

Definition 5.3. Let X be an arbitrary set and let Σ be an algebra of its subsets which contains all singletons. Suppose also that (X, Σ) is endowed with a finitely additive transition function $p(x, E)$ and $A\colon ba(X, \Sigma) \to ba(X, \Sigma)$ is the corresponding Markov operator.

Assume that $\mu_0 \in S_{ba}$ and $\mu_n = A\mu_{n-1} = A^n\mu_0$, $n = 1, 2, \dots$. By a *finitely additive Markov chain* (MC) on (X, Σ) we mean the iterative process $\{\mu_n\} = \big\{\mu_n(\mu_0)\big\}$ depending on the initial measure μ_0 as a parameter. A finitely additive MC is uniquely determined by $p(x, E)$ and μ_0. We will often identify an MC with the iterative process itself, disregarding the initial measure μ_0.

As in Section 3, we can consider finitely additive MC's on topological phase spaces $(X, \mathcal{A})$ or $(X, \mathcal{B})$. If a finitely additive MC is Feller then it is not necessarily additive. However, the following assertion holds.

Theorem 5.2. *Assume given a finitely additive Feller* MC *on a Hausdorff compact space* $(X, \mathcal{B})$, *i.e.*, $TC(X) \subset C(X)$. *Then* MC *is countably additive and* $A\colon rca(X, \mathcal{B}) \to rca(X, \mathcal{B})$; *moreover*, A *is adjoint to* $T\colon C(X) \to C(X)$.

The proof is straightforward and therefore omitted.

We now turn to the question of invariant measures for a finitely additive MC. The corresponding operator T has a fixed point $f(x) \equiv 1$ which is an interior point of K^B in $B(X, \Sigma)$ (as for Σ an algebra, as for Σ a σ-algebra). If the MC is Feller then $f(x) \equiv 1$ is also an interior fixed point of T in K^C. Hence, as in the case of countably additive MC's, the Kreĭn–Rutman theorem (see [18]) immediately implies the following assertions (Basic Theorems).

Theorem 5.3. *For every finitely additive* MC, $\Delta = \Delta_{ba} \neq \varnothing$.

Theorem 5.4. *For every Feller finitely additive* MC, $\Delta_{rba} \neq \varnothing$.

Theorem 5.5. *For every Feller finitely additive* MC *on a Hausdorff compact space,* $\Delta_{rca} \neq \varnothing$.

In view of Theorem 5.2, Theorem 5.5 is a simple rephrasing of Theorem 4.5.

In [22], Theorem 5.3 was proven only for the special case of a discrete phase space on using the technique of Banach limits.

In this article, we do not aim at a separate thorough study of finitely additive Markov chains. The constructions and theorems of this section themselves are sufficient for what follows.

6. Properties of the sets of invariant measures for the Markov operators

Let X be an arbitrary set and let Σ be a σ-algebra of its subsets which contains all singletons. Suppose that we have a countably additive MC with transition function $p(x, E)$ on (X, Σ).

Theorem 6.1. *Suppose that* $\mu \in \Delta_{ba}$, $K_\mu \in \Sigma$, *and* $\mu(K_\mu) = 1$. *Then, for arbitrary numbers* $\{\varepsilon_n\}$ *with* $0 < \varepsilon_n \leq 1$, *there exist sets* $\{K_n\}$, $K_n \in \Sigma$, $n = 1, 2, \dots$, *such that* $K_\mu \supset K_1 \supset K_2 \supset \dots$, $\mu(K_n) = 1$, *and* $p(x, K_n) \geq 1 - \varepsilon_n$ *for every* $x \in K_{n+1}$ *with* $n = 1, 2, \dots$.

Proof. Putting $\varepsilon_0 = \varepsilon_1$ and $K_0 = K_\mu$, construct the following sequence of sets for $n = 0, 1, 2, \dots$:

$$K_{n+1} \stackrel{\text{def}}{=} \{x \in K_n : p(x, K_n) \geq 1 - \varepsilon_n\}.$$

Obviously, $K_n \in \Sigma$, $n = 1, 2, \dots$, and $K_0 \supset K_1 \supset \dots$. We now proceed as follows:

$$\begin{aligned} 1 = \mu(K_0) = A\mu(K_0) = \int_X p(x, K_0)\mu(dx) = \int_{K_0} = \int_{K_1} + \int_{K_0 \setminus K_1} \\ \leq \mu(K_1) + (1 - \varepsilon_0)\mu(K_0 \setminus K_1) = 1 - \varepsilon_0 + \varepsilon_0 \mu(K_1). \end{aligned}$$

Consequently, $\mu(K_1) \geq 1$, i.e., $\mu(K_1) = 1$. Similarly, we have $\mu(K_n) = 1$, $n = 1, 2, \dots$. The theorem is proven.

Theorem 6.2. *Suppose that* $\mu \in \Delta_{ca}$, $K_\mu \in \Sigma$, *and* $\mu(K_\mu) = 1$. *Then there exists a set* $K \in \Sigma$ *such that* $K \subset K_\mu$, $\mu(K) = 1$, *and* $p(x, K) = 1$ *for every* $x \in K$, *i.e.,* K_μ *has a stochastically closed subset* K.

Proof. Putting $K_0 = K_\mu$, construct a sequence of sets for $n = 0, 1, 2, \dots$:

$$K_{n+1} \stackrel{\text{def}}{=} \{x \in K_n : p(x, K_n) = 1\}.$$

Clearly, $K_n \in \Sigma$, $n = 1, 2, \dots$, and $K_0 \supset K_1 \supset \dots$. Hence there exists a limit $K = \lim K_n = \bigcap K_n$, $K \in \Sigma$, $K \subset K_n$, $n = 1, 2, \dots$.

We now prove that $\mu(K_1) = 1$. Let $\mu(K_1) < 1$. Then $\mu(K_0 \setminus K_1) > 0$ and $p(x, K_0) < 1$ for every $x \in K_0 \setminus K_1$. Since μ is countably additive and Σ is a σ-algebra, a standard argument easily yields the inequality

$$\int_{K_0 \setminus K_1} p(x, K_0)\mu(dx) < \mu(K_0 \setminus K_1).$$

Now,

$$1 = \mu(K_0) = A\mu(K_0) = \int_X p(x, K_0)\mu(dx) = \int_{K_0} = \int_{K_1} + \int_{K_0 \setminus K_1}$$
$$< \mu(K_1) + \mu(K_0 \setminus K_1) = \mu(K_0) = 1.$$

The contradiction shows that $\mu(K_1) = 1$.

Similarly, we have $\mu(K_n) = 1$ for $n = 1, 2, \dots$. Since μ is countably additive and $\{K_n\}$ is monotone, we have

$$\mu(K) = \mu(\lim K_n) = \lim \mu(K_n) = 1.$$

Consequently, $K \neq \varnothing$. Suppose that $x \in K$. Then $x \in K_n$ and $p(x, K_{n-1}) = 1$ for all $n = 1, 2, \dots$. Since $p(x, \cdot) \in ca(X, \Sigma)$, it follows that $p(x, K) = p(x, \lim K_n) = \lim p(x, K_n) = 1$. The theorem is proven.

We now establish some structure properties of the Markov operators that will be important below. For brevity, we write $\dim \Delta$ for the dimension $\dim \operatorname{lin} \Delta$ of the linear subspace spanned by a set Δ. By a basis for Δ we also mean a basis consisting of elements of Δ in this linear subspace.

Theorem 6.3. *Assume given an* MC *on an arbitrary* (X, Σ). *If* $\dim \Delta_{ca} = n < \infty$ *then there is a basis of pairwise singular measures* $\{\mu_1, \dots, \mu_n\}$ *for* Δ_{ca}. *Moreover, there exist* $K_1, \dots, K_n \in \Sigma$ *such that* $K_i \cap K_j = \varnothing$, $i \neq j$, $\mu_i(K_i) = 1$, *and* $p(x, K_i) = 1$ *for every* $x \in K_i$, *where* $i = 1, \dots, n$.

Proof. Consider the restriction of A to $ca(X, \Sigma)$. The operator A is linear, positive, and isometric on K^{ca}. By Birkhoff's theorem (see [6, Chapter XVI, Section 7, Theorem 12]), the set of all its fixed points H is a linear subspace and a sublattice in $ca(X, \Sigma)$, i.e., for all $\mu_1, \mu_2 \in H$, $\sup(\mu_1, \mu_2)$ and $\inf(\mu_1, \mu_2)$ are also in H. By Yudin's theorem (see [28, p. 89]), there is a basis $\{\mu_1, \dots, \mu_n\}$ in H that consists of disjoint (and even discrete) elements. We may assume all $\mu_1, \dots, \mu_n$ to be positive. Normalize $\mu_1, \dots, \mu_n$, choose sets $K_{\mu_1}, \dots, K_{\mu_n} \in \Sigma$ such that $K_{\mu_i} \cap K_{\mu_j} = \varnothing$ for $i \neq j$ and $\mu_i(K_{\mu_i}) = 1$, and then apply Theorem 6.2 to each $\mu_1, \dots, \mu_n$. We have a claimed collection of sets $K_1, \dots, K_n$. The theorem is proven.

Now, we need a modification of Birkhoff's theorem mentioned above for the case of spaces of measures.

Theorem 6.4. *The sets of fixed points of the Markov operator in $ba(X,\Sigma)$ or in $ca(X,\Sigma)$ are* K*-spaces, i.e., conditionally complete vector lattices.*

Proof. Let H be the set of all fixed points of A in M, where M stands for $ba(X,\Sigma)$ or $ca(X,\Sigma)$. Clearly, H is linear. Suppose that $E \subset H \cap K^M$ and E is bounded in H. Then E is bounded in M and hence there exists $z = \sup_M E$. Since A is positive, from $z \geq x$ it follows that $Az \geq Ax = x$, $x \in E$. Therefore, $Az \geq z$. Since $z \geq 0$ and A is an isometry, we have $z = Az$, i.e., $z \in H$. Obviously, $z = \sup_H E$. This is enough (see [28, p. 92]) for H to be a K-space. The theorem is proven.

Theorem 6.5. *Suppose that an* MC *is given on an arbitrary (X,Σ). If $\dim \Delta_{ca} = \infty$ then there is a sequence $\{\mu_n\}$ of pairwise singular measures in Δ_{ca}. Moreover, there exists a sequence of measurable sets $\{K_n\}$ such that $K_n \cap K_m = \varnothing$, $n \neq m$, $\mu_n(K_n) = 1$, and $p(x, K_n) = 1$ for every $x \in K_n$ with $n = 1, 2, \dots$.*

Proof. In accordance with Theorem 6.4., the set H of all fixed points of A in $ca(X,\Sigma)$ is an infinite-dimensional K-space. It is known that every infinite-dimensional K-space has an infinite set of pairwise disjoint elements. For pairs of measures, disjointness means singularity, which implies existence of disjoint sets of full measure for countably additive measures. From this fact and Theorem 6.2 the claim of the theorem follows.

We now return to the general case of Theorem 6.1. The difference between this theorem and the special case of countably additive invariant measures is clear. In the following theorem, we specify the properties of a sequence of "supports" of invariant measures in the finitely additive case.

Theorem 6.6. *Suppose that $\mu \in \Delta_{ba}$ and a sequence $\{M_n\}$, where $M_n \in \Sigma$, $M_1 \supset M_2 \supset \dots$, and $\lim M_n = \bigcap M_n = \varnothing$, meets the equality $\mu(M_n) = 1$ for all $n \in \mathbb{N}$. Then, for arbitrary numbers $\{\varepsilon_n\}$ with $0 < \varepsilon_n \leq 1$, $n = 1, 2, \dots$, there exist sets $\{K_n\}$, $K_n \in \Sigma$, $n = 1, 2, \dots$, such that $K_1 \supset K_2 \supset \dots$, $\lim K_n = \varnothing$, $\mu(K_n) = 1$, $p(x, K_n) \geq 1 - \varepsilon_n$ for every $x \in K_{n+1}$ for all $n = 1, 2, \dots$.*

Proof. Repeating the scheme of the proof of Theorem 6.1, put $K_1 = M_1$ and construct a sequence of sets

$$K_{n+1} \overset{\text{def}}{=} \big\{x \in K_n : p(x, K_n) \geq 1 - \varepsilon_n\big\} \cap M_{n+1}, \quad n \in \mathbb{N}.$$

Clearly, $K_n \in \Sigma$, $n \in \mathbb{N}$, and $K_1 \supset K_2 \supset \dots$. Since $K_n \subset M_n$, we have $\lim K_n = \bigcap K_n = \varnothing$.

Making the same integral transformations as in the proof of Theorem 6.1, we have $\mu(K_n) = 1$, $n = 1, 2, \dots$. The theorem is proven.

The conditions of Theorem 6.6 are not burdensome. If a measure λ is not countably additive then, by definition, there always exists a sequence of sets $K_1 \supset K_2 \supset \dots,\ K_n \to \varnothing$, such that $\lim \lambda(K_n) > 0$, i.e., the inequality $\lambda(K_n) \geq \rho$ holds for some $\rho > 0$ and all $n \in \mathbb{N}$.

The measure μ *in Theorem* 6.6 *is automatically purely finitely additive.*

It is impossible to state more, i.e., to distinguish stochastically closed sets in Theorem 6.6 (and in Theorem 6.1) without imposing extra conditions on the invariant measure. This is exemplified as follows.

Example 6.1. Assume that $X = [0,1]$, $\Sigma = \mathcal{B}$, and the Markov chain is as follows: from $x \in (0,1]$, there is a transition to x^2 with probability x and to 0, with probability $1 - x$, whereas 0 is a stationary point. Formally, this means that $p(x, E) = x\delta_{x^2}(E) + (1-x)\delta_0(E)$ if $x \neq 0$ and $p(0,E) = \delta_0(E)$. Here we do not perform easy but rather long calculations to show that the MC has an invariant purely finitely additive measure λ "near 1," i.e., λ satisfies the condition $\lambda\big((1-\varepsilon, 1)\big) = 1$ for every $0 < \varepsilon < 1$. Moreover, any set $K \in \Sigma$, say, in $\left(\frac{1}{2}, 1\right)$, is not stochastically closed for the chain and yet the assertion of Theorem 6.6 holds.

In Section 7, we give a conversion of Theorem 6.6.

7. Weak limit points of Cesàro means and invariant measures

In this section, we consider Cesàro means for Markov sequences. In the sequel, we will be especially interested in the limit behavior of means in the topologies generated by spaces of functions in spaces of measures. All topological spaces below are assumed normal.

We denote the *Cesàro means* for a measure μ as follows:

$$\lambda_n = \lambda_n^\mu = \frac{1}{n}\sum_{k=1}^{n} A^k \mu, \quad n \in \mathbb{N}.$$

Theorem 7.1. *Suppose that* X *is normal and we have a Feller* MC $\mu \in rba(X, \mathcal{A})$, $\mu \in S_{rba}$ *on* $(X, \mathcal{B})$. *Then every* τ_C*-limit measure of* $\{\lambda_n^\mu\}$ *in* $rba(X, \mathcal{A})$ *is a fixed point of* A, *i.e.,* $\mathfrak{N}\{\lambda_n\} \subset \Delta_{rba}$, *and the set of such measures is nonempty, i.e.,* $\mathfrak{N}\{\lambda_n\} \neq \varnothing$, *and* τ_C*-compact.*

Proof. Choose $\mu \in S_{rba}$. Obviously, $\|\lambda_n\| = \lambda_n(X) = 1$, $n = 1, 2, \dots$, i.e., the set $\{\lambda_n\}$ is metrically bounded in $rba(X, \mathcal{A})$. Consequently, the τ_C-closure $\{\lambda_n\}$ is compact in the τ_C-topology (see [9, Chapter V, Item 4, Corollary 3]). According to [17, Chapter V, Theorem 5], every subsequence λ_{n_i} including λ_n has a limit point $\eta = \eta\{\lambda_{n_i}\}$ such that each of its neighborhoods contains infinitely many elements of the subsequence. This means that, for every τ_C-neighborhood $V(\eta, f_1, f_2, \dots, f_k, \varepsilon)$, the set $\big\{i : \lambda_{n_i} \in V(\eta, f_1, f_2, \dots, f_k, \varepsilon)\big\}$ is infinite, i.e., there exists a subsequence $\{\lambda_{n_{i_j}}\}$, $\lambda_{n_{i_j}} \in V(\eta, f_1, f_2, \dots, f_k, \varepsilon)$, $j = 1, 2, \dots$.

Suppose that η is τ_C-limit for $\{\lambda_n\}$. We proceed as follows:

$$A\lambda_n = \frac{1}{n}\sum_{k=1}^{n} A^k\mu + \frac{1}{n}[A^{n+1}\mu - A\mu] = \lambda_n + \frac{1}{n}[A^{n+1}\mu - A\mu].$$

Assume that $f \in C(X)$. Then, for every $\varepsilon > 0$, there exists a strictly increasing sequence of numbers $\{n_i\} = \{n_i\}(f,\varepsilon)$ such that $\lambda_{n_i} \in V(\eta, f, Tf, \varepsilon)$, $i = 1, 2, \dots$. (By hypothesis, $Tf \in C(X)$.)

Furthermore,

$$\begin{aligned}
\big|f(\eta) - f(A\eta)\big| &= \big|f(\eta) - Tf(\eta)\big| \\
&\le \big|f(\eta) - f(\lambda_{n_i})\big| + \big|f(\lambda_{n_i}) - Tf(\lambda_{n_i})\big| + \big|Tf(\lambda_{n_i}) - Tf(\eta)\big| \\
&\le \varepsilon + \big|f(\lambda_{n_i}) - f(A\lambda_{n_i})\big| + \varepsilon \\
&= 2\varepsilon + \left|f(\lambda_{n_i}) - f(\lambda_{n_i}) - f\left(\frac{1}{n_i}[A^{n_i+1}\mu - A\mu]\right)\right| \\
&\le 2\varepsilon + \frac{1}{n_i}\big|f(A^{n_i+1}\mu - A\mu)\big| \\
&\le 2\varepsilon + \frac{2\|f\|}{n_i}.
\end{aligned}$$

Since $n_i \to \infty$ as $i \to \infty$, we have $\big|f(\eta) - f(A\eta)\big| \le 2\varepsilon$. So, by arbitrariness of ε, we infer that $\big|f(\eta) - f(A\eta)\big| = 0$.

Thus, each $f \in C(X)$ meets the equality $f(\eta) = f(A\eta)$. The set $C(X)$ is total on $rba(X, \mathcal{A})$; therefore, $\eta = A\eta$. Moreover, $\eta \in rba(X, \mathcal{A})$.

Now, prove that $\eta \in S_{rba}$, i.e., that η is normalized and positive. Consider a τ_C-neighborhood of η of the form $V(\eta, f, \varepsilon)$, where $\varepsilon > 0$ is arbitrary, $f \in C(X)$, and $f(x) \equiv 1$. Then there exists an n_i such that $\lambda_{n_i} \in V(\eta, f, \varepsilon)$, i.e.,

$$\big|\langle f, \eta\rangle - \langle f, \lambda_{n_i}\rangle\big| = \big|\eta(X) - \lambda_{n_i}(X)\big| < \varepsilon.$$

The equality $\lambda_n(X) = 1$ for all $n \in \mathbb{N}$ implies $\big|\eta(X) - 1\big| < \varepsilon$ for every $\varepsilon > 0$, whence $\eta(X) = 1$.

Suppose that there exists $E \in \mathcal{A}$ such that $\eta(E) = -r < 0$. Since η is regular, for every $\varepsilon > 0$, we can find sets $F = \overline{F} \in \mathcal{A}$ and $G = \mathring{G} \in \mathcal{A}$ such that $F \subset E \subset G$ and $-r - \varepsilon < \eta(G) \le \eta(E) \le \eta(F) < -r + \varepsilon$. Choose $\varepsilon = \frac{r}{2}$. Then we have

$$-\frac{3r}{2} < \eta(G) \le \eta(F) \le -\frac{r}{2}$$

for appropriate F and G.

Since X is normal and $X \setminus G$ and F are its disjoint closed subspaces, it follows that, by the famous Urysohn theorem ([9, Chapter I, Item 5, Theorem 2]), there exists $f \in C(X)$ such that $0 \le f(x) \le 1$, $f(X \setminus G) = 0$, and $f(F) = 1$. Omitting simple estimations of integrals, we have

$$\big|\langle f, \lambda_n\rangle - \langle f, \eta\rangle\big| \ge \lambda_n(F) + \frac{r}{2} \ge \frac{r}{2} > 0$$

for all $n \in \mathbb{N}$.

Consequently, for $\varepsilon < \frac{r}{2}$, there is no $n \in \mathbb{N}$ such that $\lambda_n \in V(\eta, f, \varepsilon)$, i.e., the measure η is not τ_C-limit for $\{\lambda_n\}$. The contradiction shows that $\eta(E) \ge 0$ for all $E \in \mathcal{A}$. Thus $\eta \in S_{rba}$ and $\eta \in \Delta_{rba}$, i.e., $\mathfrak{N}\{\lambda_n\} \subset \Delta_{rba}$. The theorem is proven.

If η is a limit point for $\{\lambda_n\}$ in τ_C then there need not be a sequence τ_C-converging to η.

If $\mu \in rca(X, \mathcal{B})$ then $\lambda_n \in rca(X, \mathcal{B})$, $n = 1, 2, \dots$. Generally speaking, a τ_C-limit measure for such $\{\lambda_n\}$ is not countably additive.

Theorem 7.2. *Suppose that we have an arbitrary* MC *on an arbitrary* (X, Σ), $\mu \in ba(X, \Sigma)$, *and* $\mu \in S_{ba}$. *Then every* τ_B*-limit point of the sequence* $\{\lambda_n^\mu\}$ *in* $ba(X, \Sigma)$ *is a fixed point of* A, *i.e.,* $\mathfrak{M}\{\lambda_n\} \subset \Delta_{ba}$, *the set of such measures is nonempty, i.e.,* $\mathfrak{M}\{\lambda_n\} \ne \varnothing$, *and* τ_B*-compact.*

Proof. Since $B^* = ba(X, \Sigma)$, we should just repeat the first part of the proof of Theorem 7.1 replacing C by B and rba, by ba. The fact that the τ_C-limit measures are normed and positive is proven even easier here. The theorem is proven.

Corollary 7.1. *All the* τ_C*- and* τ_B*-limit measures* η *in Theorems* 7.1 *and* 7.2 *respectively meet the conditions* $0 \le \eta$ *and* $\|\eta\| = \eta(X) = 1$.

If the MC on $(X, \mathcal{B})$ is not Feller then all the τ_B-limit measures of $\{\lambda_n\}$ are invariant but the τ_C-limit measures (there are more of them, and they include all τ_B-limit measures) need not be invariant, i.e., $\mathfrak{M}\{\lambda_n\} \subset \Delta_{ba}$ but $\mathfrak{N}\{\lambda_n\} \not\subset \Delta_{rba}$.

We point out that, in the following two theorems, the means are taken for an MC having a different initial measure μ_n for each $n \in \mathbb{N}$. This is done for further applications rather than for the sake of generalization.

Theorem 7.3. *Suppose that we have an arbitrary* MC *on an arbitrary* (X, Σ), $\mu_n \in ba(X, \Sigma)$, $\mu_n \in S_{ba}$, *and*

$$\lambda_n = \lambda_n^{\mu_n} = \frac{1}{n}\sum_{k=1}^{n} A^k \mu_n, \quad n = 1, 2, \dots .$$

Then each τ_B*-limit measure of* $\{\lambda_n\}$ *is invariant for* A, *i.e.,* $\mathfrak{M}\{\lambda_n\} \subset \Delta_{ba}$, *the set of such measures is nonempty and* τ_B*-compact.*

Proof. We have

$$A\lambda_n = \frac{1}{n}\sum_{k=1}^{n} A^k\mu_n + \frac{1}{n}[A^{n+1}\mu_n - A\mu_n] = \lambda_n + \frac{1}{n}[A^{n+1}\mu_n - A\mu_n].$$

Now, following the scheme of the proof of Theorem 7.1, we obtain the desired.

Remark. Obviously, if $\eta \in \mathfrak{M}(\lambda_n)$ then $\eta(E) = \lim \lambda_n(E)$ for all $E \in \Sigma$ for which $\lim \lambda_n(E)$ is defined.

Theorem 7.4. *Suppose that we have a Feller* MC *on a topological phase space* $(X, \mathcal{A})$, $\mu_n \in rba(X, \mathcal{A})$, $\mu_n \in S_{rba}$, *and*

$$\lambda_n = \lambda_n^{\mu_n} = \frac{1}{n}\sum_{k=1}^{n} A^k\mu_n, \quad n = 1, 2, \dots .$$

Then every τ_C*-limit point of* $\{\lambda_n\}$ *is invariant for* A, *i.e.,* $\mathfrak{N}\{\lambda_n\} \subset \Delta_{rba}$, *the set of such measures is nonempty and* τ_C*-compact.*

The proof is carried out in much the same way as that of Theorem 7.3.

Corollary 7.2. *Theorems* 7.1–7.4 *remain valid on substituting arbitrary subsequences* $\{\lambda_{n_i}\}$ *for* $\{\lambda_n\}$.

Indeed, it is for an arbitrary subsequence $\{\lambda_{n_i}\}$ that Theorem 7.1 is actually proven, and the proofs of the other theorems repeat its proof.

In the following theorems, X is assumed to be a normal topological space. Recall that the C-equivalence class for $\mu \in rba(X, \Sigma)$ is the set $\mathcal{R}\{\mu\} = \big\{\lambda \in ba(X, \Sigma) : \lambda \geq 0,\ \overline{\lambda} = \mu\big\}$, where $\overline{\lambda}$ is the regularization of λ.

Theorem 7.5. *Assume given a Feller* MC *on* $(X, \mathcal{B})$ *and* $\mu = A\mu \in S_{ba}$. *Then* $\bar{\mu} = A\bar{\mu}$.

Proof. By definition, we have

$$f(\bar{\mu}) = f(\mu) = f(A\mu) = Tf(\mu) = Tf(\bar{\mu}) = f(A\bar{\mu})$$

for every $f \in C(X)$. Since $C(X)$ is total on $rba(X, \mathcal{A})$, it follows that $\bar{\mu} = A\bar{\mu}$. The theorem is proven.

Theorem 7.6. *Assume given a Feller* MC *on* $(X, \mathcal{B})$ *and* $\mu = A\mu \in S_{rba}$. *Then* $A\big(\mathcal{R}\{\mu\}\big) \subset \mathcal{R}\{\mu\}$.

Proof. Assume that $\lambda \in \mathcal{R}\{\mu\}$, i.e., $f(\lambda) = f(\mu)$ for every $f \in C(X)$. Then $f(A\lambda) = Tf(\lambda) = Tf(\mu) = f(A\mu) = f(\mu)$ for $f \in C(X)$, i.e., $A\lambda \in \mathcal{R}\{\mu\}$. The theorem is proven.

Theorem 7.7. *Let* X *satisfy the condition* $ca(X, \mathcal{B}) = rca(X, \mathcal{B})$, *and we have an arbitrary* MC *on* $(X, \mathcal{B})$. *If* $\mu \in rca(X, \mathcal{B})$ *is such that* $A\mathcal{R}\{\mu\} \subset \mathcal{R}\{\mu\}$ *then* $\mu = A\mu$.

Proof. Since $Aca(X, \mathcal{B}) \subset ca(X, \mathcal{B})$, we have $A\mu \in ca(X, \mathcal{B}) \cap \mathcal{R}\{\mu\}$. However, $\mathcal{R}\{\mu\}$ contains only one countably additive measure. The theorem is proven.

Now, we are able to give the announced conversion of Theorem 6.6. First, we prove the following lemma.

Lemma 7.1. *Assume that X is arbitrary, Σ is a σ-algebra of subsets of X, $K_n \in \Sigma$, $K_n \neq \varnothing$ for $n \in \mathbb{N}$, $K_1 \supset K_2 \supset \dots$, and $\bigcap K_n = \varnothing$. Then there exists a purely finitely additive measure $\lambda \in S_{ba}$ such that $\lambda(K_n) = 1$ for all $n \in \mathbb{N}$.*

Proof. Suppose that $x_1 \in K_1$. Then there exists $n_1 > 1$ such that $x_1 \notin K_{n_1}$ $\big($otherwise $x_1 \in \bigcap K_n = \varnothing\big)$. Choose $x_2 \in K_{n_1}$. Obviously, $x_1 \neq x_2$. There exists $n_2 > n_1$ such that $x_2 \notin K_{n_2}$, etc. We obtain a sequence $x_n \in K_1$, $n \in \mathbb{N}$. Put $M_1 = \{x_1, x_2, x_3, \dots\}$, $M_2 = \{x_2, x_3, x_4, \dots\}$, etc. We have

$$K_1 \supset M_1, \quad K_2 \supset M_2 \supset \dots, \quad M_1 \supset M_2 \supset \dots, \quad \bigcap_{n=1}^{\infty} M_n = \varnothing.$$

For the countable family of sets M_n, there exists a purely finitely additive measure $\lambda \in S_{ba}$ such that $\lambda(M_1) = \lambda(M_2) = \dots = 1$. Extend λ to the whole X by zero outside M_1; then $\lambda(K_n) = 1$ for all $n \in \mathbb{N}$. The lemma is proven.

Theorem 7.8. *Suppose that we have an* MC *on an arbitrary (X, Σ) and that there exist sequences ε_n and K_n such that*

$$\varepsilon_n \geq 0, \quad \varepsilon_n \to 0 \ \text{ as } \ n \to \infty,$$

$$K_n \in \Sigma, \quad K_n \neq \varnothing \ \text{ for } \ n \in \mathbb{N}, \quad K_1 \supset K_2 \supset \dots, \quad \bigcap_{n=1}^{\infty} K_n = \varnothing,$$

and

$$p(x, K_n) \geq 1 - \varepsilon_n \ \text{ for } \ x \in K_{n+1}, \quad n \in \mathbb{N}.$$

Then the MC *has an invariant purely finitely additive measure $\mu \in \Delta_{ba}$; moreover, $\mu(K_n) = 1$ for $n = 1, 2, \dots$.*

Proof. Using Lemma 7.1, choose an arbitrary purely finitely additive measure $\lambda \in S_{ba}$ such that $\lambda(K_n)$=1 for $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$A\lambda(K_n) = \int_X p(x, K_n)\lambda(dx) \geq \int_{K_{n+1}} p(x, K_n)\lambda(dx) \geq (1-\varepsilon_n)\lambda(K_{n+1}) = 1-\varepsilon_n.$$

Since $K_1 \supset K_n$ for all $n \in \mathbb{N}$, we have $A\lambda(K_1) \geq A\lambda(K_n) \geq (1 - \varepsilon_n)$, where $1 - \varepsilon_n \to 1$ as $n \to \infty$. Consequently, $A\lambda(K_1) = 1$. Similarly, we have

$A\lambda(K_n) = 1$ for $n \in \mathbb{N}$. Repeating the argument for the measure $\lambda_1 = A\lambda_1$, we infer that $A\lambda_1(K_n) = A^2\lambda(K_n) = 1$ for $n \in \mathbb{N}$. Analogously, we conclude that $A^m\lambda(K_n) = 1$ for all $n, m \in \mathbb{N}$. Hence we have

$$\frac{1}{n}\sum_{m=1}^{n} A^m\lambda(K_i) \equiv 1 \quad \text{for } n \in \mathbb{N}$$

for every $i \in \mathbb{N}$.

Let μ be a τ_B-limit measure for the sequence

$$\frac{1}{n}\sum_{m=1}^{n} A^m\lambda.$$

By Theorem 7.2, such a measure exists and is invariant for MC, i.e., $\mu \in \Delta_{ba}$. Then the equality

$$\mu(E) = \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} A^m\lambda(E)$$

holds for those $E \in \Sigma$ for which the limit exists. Hence $\mu(K_i) = 1$ for $i \in \mathbb{N}$. The relations $\bigcap K_i = \varnothing$ imply that μ is not countably additive because

$$1 = \lim_{i\to\infty}\mu(K_i) \neq \mu\Big(\lim_{i\to\infty} K_i\Big) = \mu(\varnothing) = 0.$$

Let $\mu = \mu_1 + \mu_2$ be the decomposition of μ into purely finitely additive and countably additive components μ_1 and μ_2. The above implies that $\mu_1 \neq 0$. Moreover, as already noted, $\mu_1 = A\mu_1$ and $\mu_2 = A\mu_2$. The countably additive measure μ_2 must satisfy $\mu_2(K_i) \to 0$ as $i \to \infty$. Consequently, $\mu_1(K_i) \to 1$ as $i \to \infty$. Since $\mu_1 \geq 0$ and $K_1 \supset K_2 \supset \dots$, we have $\mu_1(K_i) = 1$ for all $i \in \mathbb{N}$. We have thus obtained a desired purely finitely additive measure. The theorem is proven.

8. Dimension of the set of invariant measures

Although this section is short, it contains assertions on possible dimensions of sets of invariant measures for an arbitrary MC which will be important in the sequel.

Theorem 8.1. *Suppose that an* MC *is given on an arbitrary* (X, Σ). *If* $\dim\Delta_{ca} = \infty$ *then* $\Delta_{pfa} \neq \varnothing$ *and* $\dim\Delta_{pfa} = \infty$.

Proof. Suppose that $\dim\Delta_{ca} = \infty$. By Theorem 6.5, there exist a sequence $\{\mu_n\}$ of pairwise singular measures in Δ_{ca} and a sequence $\{K_n\}$ of measurable sets such that $K_n \cap K_m = \varnothing$ for $n \neq m$ and $\mu_n(K_n) = 1$ for $n = 1, 2, \dots$.

Let $\mathfrak{M}$ be the set of all τ_B-limit measures of $\{\mu_n\}$. Clearly, $\mathfrak{M} \neq \varnothing$ and $\mathfrak{M} \subset \Delta$. By Theorem 2.7, we have $\mathfrak{M} \subset L\{\mu_n\}$, and, by Theorem 2.2′, all Banach limits in $L\{\mu_n\}$ are purely finitely additive. Therefore, $\Delta_{pfa} \neq \varnothing$.

Split $\{K_n\}$ into a countable number of mutually disjoint subsequences $\{K_{n_1}, K_{n_2}, \dots\}$ to which there correspond subsequences of measures $\xi = \{\mu_{n_1}, \mu_{n_2}, \dots\}$. Every τ_B-limit measure μ_ξ for such a sequence ξ belongs to $\mathfrak{M}$ and $\mu_\xi\bigl(\bigcup K_{n_i}\bigr) = 1$. Obviously, for different ξ, their τ_B-limit measures μ_ξ are singular. Thus, for the whole sequence $\{\mu_n\}$, there exists a countable set $\{\mu_\xi\}$ of pairwise singular τ_B-limit measures; moreover, $\mu_\xi \in \Delta_{pfa}$. The set $\{\mu_\xi\}$ is linearly independent and, hence, $\dim \Delta_{pfa} = \infty$. The theorem is proven.

The following assertion is a direct corollary to Theorem 8.1 but we also call it a theorem to stress its importance in what follows.

Theorem 8.2. *Assume given an* MC *on an arbitrary* (X, Σ). *If* $\Delta \subset ca(X, \Sigma)$ *then* $\dim \Delta < \infty$.

We now formulate the converse of this theorem for $n = 1$.

Theorem 8.3. *Assume given an* MC *on an arbitrary* (X, Σ). *If* $\dim \Delta = 1$, *i.e., if the* MC *has a unique invariant measure* μ *in* S_{ba}, *then* $\Delta \subset ca(X, \Sigma)$, *i.e.,* μ *is countably additive. Moreover,* $\lambda_n^\eta \to \mu$ *in* τ_B *for every* $\eta \in S_{ba}$.

Proof. Take $\eta \in S_{ba}$. Let $\mathfrak{M}$ be the set of all τ_B-limit measures of $\{\lambda_n^\eta\}$. By Theorem 7.2, we have $\mathfrak{M} \neq \varnothing$ and $\mathfrak{M} \subset \Delta$. This and the hypothesis of the theorem imply that $\mathfrak{M} = \Delta = \{\mu\}$, i.e., $\{\lambda_n^\eta\}$ has a unique τ_B-limit point.

We now demonstrate that $\lambda_n^\eta \to \mu$ in τ_B. Suppose the contrary, i.e., that there exist $E \in \Sigma$, $\delta > 0$, and $\{n_i\}$ such that $\bigl|\lambda_{n_i}^\eta(E) - \mu(E)\bigr| \geq \delta$, $i = 1, 2, \dots$. Then $\{\lambda_{n_i}^\eta\}$ has a τ_B-limit point $\xi \neq \mu$. We know that all τ_B-limit points of every subsequence $\{\lambda_{n_i}^\eta\}$ are τ_B-limit for $\{\lambda_n^\eta\}$ itself, i.e., $\xi \in \mathfrak{M} = \{\mu\}$. The contradiction shows that $\lambda_n^\eta \to \mu$ in τ_B.

Suppose now that η is countably additive, i.e., $\eta \in S_{ca}$. Since all λ_n^η, $n = 1, 2, \dots$, are countably additive, by the Nikodým theorem ([9, Chapter III, Item 7, Corollary 4]), μ is countably additive too. The theorem is proven.

In Section 12 (in the second part of the article), we will prove that the condition $\Delta \subset ca(X, \Sigma)$ is equivalent to the well-known Doob–Doeblin Condition.

3. MARKOV CHAINS ON THE GAMMA-COMPACTIFICATION OF A MEASURE SPACE

9. Gamma-compactification of a measure space

In the study of Markov chains, it is necessary to consider trajectories tending to the “boundary” of the phase space (X, Σ). Since X need not have a “natural” boundary as, for example, in the case of the real line, we would like to close X somehow, or, better, to embed it into a compact space. In such a compactification, it is possible to fix the (adjoined) points to which the trajectories of the Markov chains converge. Explicit procedures of such extensions of the phase space for various stochastic processes were developed by many authors. In [11, 25], some stochastic processes are extended to the Stone–Čech compactification βX of the initial topological phase space X, and, in [19], its one-point compactification is considered. A special type of extension of the phase space is suggested in [20] and developed in [7] for chains irreducible in the sense of Harris.

The instrument of compactifications of a topological space, suitable for these purposes, is well developed in general topology. All compactifications bX considered by topologists are between the *minimal* Alexandrov one-point compactification αX and *maximal* Stone–Čech compactification βX. Maximality of βX means that βX is maximal among the compactifications bX of X for which X is *homeomorphically* embeddable into bX. Topologists do not consider nonhomeomorphic embeddings of X into compact spaces; therefore, traditional “users” of the theory also never involve such compactifications. However, there are other compactifications of the initial space which we will just have to deal with because we also consider nontopological X's.

As a particular desired extension of (X, Σ), we take the space of maximal ideals of the Banach algebra $B(X, \Sigma)$. Unfortunately, this space was not studied deeply enough in Gel'fand's theory of Banach algebras (rings), which gave rise to the author's research on the topic (see [31, 32, 34]). In order not to complicate considering the ergodic theorems in the article, in this section, we briefly describe some features of the construction and only point out the topic of the author's research, which is unavoidable in what follows.

Definition 9.1. Assume given a measure space (X, Σ) whose algebra Σ contains all singletons. Define the *gamma-compactification* (*γ-compactification*) of (X, Σ) to be the set $\gamma_\Sigma X = \gamma X$ of all maximal ideals of the Banach algebra $B(X, \Sigma)$ with the Tychonoff topology, or equivalently, the set of all multiplicative functionals in $B^*(X, \Sigma)$ in the *-weak topology. We denote the topology on γX by $\tau_\gamma = \tau_{\gamma X}$.

We observe that, for an infinite nondiscrete topological space X, γX is strictly greater than the compactification βX, called maximal in general topology. However, the extension γX can be described using homeomorphic embeddings even for nontopological X's (see [26]). To this end, endow X with the discrete topology τ_0. Then consider the Wallman–Shanin compactification $w_\Sigma(X, \tau_0)$ of (X, τ_0) (see [4]) generated by the class Σ, which is a lattice of closed sets in (X, τ_0). It can be proven that $w_\Sigma(X, \tau_0) \leq \beta(X, \tau_0)$ and the embedding $(X, \tau_0) \to w_\Sigma(X, \tau_0)$ is homeomorphic. Moreover,

$B(X,\Sigma)$ is isometrically isomorphic to $C\big(w_\Sigma(X,\tau_0)\big)$ and $B^*(x,\Sigma) = ba(X,\Sigma)$, to $C^*\big(w_\Sigma(X,\tau_0)\big) = rca\big(w_\Sigma(X,\tau_0),\mathcal{B}_w\big)$, where $\mathcal{B}_w$ is the Borel σ-algebra in $w_\Sigma(X,\tau_0)$. Thus the Wallman–Shanin compactification $w_\Sigma(X,\tau_0)$ is the desired compactification γX.

The Wallman compactification wX (in its initial definition) for perfectly normal spaces X was used by A. D. Alexandrov in [1] to extend regular finitely additive measures on $(X,\mathcal{A})$ to regular countably additive measures on $(wX,\mathcal{B}_w)$.

In the general case, in defining γX as the Wallman–Shanin compactification $w_\Sigma(X,\tau_0)$, the initial topology in X ($\Sigma = \mathcal{B}$) or even the structure of Σ is "hidden" far behind the "intermediate" discrete topology τ_0. This makes the topological approach uncomfortable for our purposes. Therefore, we use the possibility of constructing the compact extension γX in the framework of Gel'fand's theory of Banach algebras.

The space $\gamma_\Sigma X$ was applied to the study of finitely additive measures in Yosida and Hewitt's article [29] (whose results we constantly use).

Let (X,Σ) be a measure space and let $\gamma_\Sigma X = \gamma X$ be its gamma-compactification. First, we give a minimal necessary information on the construction of γX, in which we are guided by [9, Chapter IV, Item 9], where the isometric isomorphism $r\colon B(X,\Sigma) \to C(\gamma X)$ is accompanied by two natural mappings.

The *first* mapping is a dense injective embedding $s\colon X \to \gamma X$ that can be defined in various models. It is convenient to represent γX as the space of all multiplicative functionals in $B^*(X,\Sigma) = ba(X,\Sigma)$. This space is known to be in a one-to-one correspondence with the class of all two-dimensional measures in $ba(X,\Sigma)$, one part of which, M_1, consists of countably additive measures and the remainder, M_2, of purely finitely additive measures. The measures of M_1 are Dirac measures δ_x with singleton atoms at points $x \in X$. The measures of M_2 are not like these and are degenerate at no point in X. In this representation, to every point $x \in X$, the mapping $s\colon X \to \gamma X \sim M = M_1 \cup M_2$ assigns injectively the Dirac measure $\delta_x \in M_1$ degenerate at x, i.e., $s(x) = \delta_x \in ba(X,\Sigma)$. Moreover, $s(X) = M_1 \subset M \sim \gamma X$. In the *-weak topology of $B^*(X,\Sigma) = ba(X,\Sigma)$ (i.e., in the τ_γ-topology of γX), M_1 is dense in M, i.e., $\overline{s(X)} = \overline{M_1} = M \sim \gamma X$.

We often identify the points $s(x)$ and x for $x \in X$. By $s(E)$ we mean the image of $E \in \Sigma$ under the pointwise mapping s. If X is a discrete topological space, i.e., all singletons in X are clopen $\big($in this case $\tau_X = \mathcal{B}_X = 2^X\big)$ then s is continuous and $\gamma X{=}\beta X$. If X is a nondiscrete topological space and $\Sigma = \mathcal{B}$ then $s\colon X \to \gamma X$ is discontinuous and γX is strictly greater than βX.

Recall that the isometry $r\colon B(X,\Sigma) \to C(\gamma X)$ is also an algebraic isomorphism, i.e., $r(f_1+f_2) = r(f_1)+r(f_2)$ and $r(f_1f_2) = r(f_1)r(f_2)$. In particular, $r(f^2) = \big[r(f)\big]^2$. Suppose that $E \in \Sigma$ and χ_E is the characteristic function of E. Then $r\big([\chi_E]^2\big) = \big[r(\chi_E)\big]^2 = r(\chi_E)$. Hence $r(\chi_E)$ may take only two values 0 and 1, i.e., $r(\chi_E) \in C(\gamma X)$ is also the characteristic function of a set $E_1 \subset \gamma X$, $r(\chi_E) = \chi_{E_1}$. The function χ_{E_1} can be continuous only if E_1 is clopen in γX. Therefore, r generates the *second* mapping t from sets $E \in \Sigma$ in X to the class of clopen sets in $\mathcal{N}_{\gamma X}$; furthermore, $r(\chi_E) = \chi_{t(E)}$ for all $E \in \Sigma$.

In [9, Chapter IV, Item 9, Lemma 10], it is proven that t is an algebraic isomorphism from the σ-algebra (or algebra) Σ onto the whole class $\mathcal{N}_{\gamma X}$, which turns out to be an algebra in γX and, moreover, a base of the topology τ_γ in γX. Thus the gamma-compactification $(\gamma X, \tau_\gamma)$ is a totally disconnected space.

Definition 9.2 [27]**.** Let (X, τ) be a topological space. Define the class of Z-sets as follows:

$$\mathcal{Z} = \big\{Z \subset X : Z = f^{-1}(0) \text{ for some } f \in C(X)\big\}.$$

We call the algebra $\mathcal{A}_Z$ and σ-algebra $\mathcal{B}_Z = \mathcal{B}_Z(X)$ generated by $\mathcal{Z}$ the *Baire algebra and σ-algebra* and refer to sets of $\mathcal{B}_Z$ as *Baire sets*.

For metric and some other spaces, the Baire σ-algebra $\mathcal{B}_Z$ coincides with the Borel σ-algebra $\mathcal{B}$. In the general case, $\mathcal{B}_Z \subset \mathcal{B}$, and the inclusion here may be strict. It is known (see [16, Chapter X, Section 51]) that, on a Hausdorff compact totally disconnected space, the Baire σ-algebra $\mathcal{B}_Z$ is generated by the algebra $\mathcal{N}$ of clopen sets, i.e., $\mathcal{B}_Z = \sigma(\mathcal{N})$. In particular, in our case, $\mathcal{B}_Z(\gamma X) = \sigma(\mathcal{N}_{\gamma X})$. Except for trivial cases, in such spaces, the Baire σ-algebra is strictly less than the Borel algebra. Note that, generally speaking, $\mathcal{N}_{\gamma X}$ is not a σ-algebra even if Σ is a σ-algebra.

For dual spaces of functions, we have the isomorphisms

$$B^*(X, \Sigma) = ba(X, \Sigma), \quad C^*(\gamma X) = rca\big(\gamma X, \mathcal{B}_{\gamma X}\big).$$

Adjoint to the isomorphism $r\colon B(X, \Sigma) \to C(\gamma X)$ is the isomorphism $r^*\colon C^*(\gamma X) \to B^*(X, \Sigma)$, i.e., $r^*\colon rca(\gamma X, \mathcal{B}_{\gamma X}) \to ba(X, \Sigma)$. Consequently, the isomorphism $[r^*]^{-1}$ gives unique extensions of finitely additive measures $\mu \in ba(X, \Sigma)$ from (X, Σ) to regular countably additive measures $\widetilde{\mu} = [r^*]^{-1}\mu \in rca(\gamma X, \mathcal{B}_{\gamma X})$ on $(\gamma X, \mathcal{B}_{\gamma X})$. In [9], it was proven that $\widetilde{\mu}(E) = \mu\big(t^{-1}(E)\big)$ for $E \in \mathcal{N}_{\gamma X}$ and $\widetilde{\mu}(tG) = \mu(G)$ for $G \in \Sigma$. There exists a unique extension of the measure $\widetilde{\mu}$, already defined on $\mathcal{N}_{\gamma X}$, to a regular countably additive measure on the Baire σ-algebra $\mathcal{B}_Z$ and the Borel σ-algebra $\mathcal{B}_{\gamma X}$.

Below, we use this information without further references.

New problems of γ-compactification arise in attempts to describe the excrescence $\gamma X \setminus X$ and understand how finitely additive measures extend to regular countably additive measures on $(\gamma X, \mathcal{B}_{\gamma X})$ exactly.

The properties of measures extended to the Stone–Čech compactification βX, under various conditions, were studied, for example, in [26, 27]. The excrescence $\beta X \setminus X$ is studied in many topological articles. The author solved these problems for γX in [31, 32, 34]. We observe that the situation with γX is essentially different from the case of βX. We will make the relevant references to the above-mentioned articles directly in the sequel.

10. The construction of the Feller extension of an arbitrary Markov chain to the gamma-compactification of the phase space

Let X be a set and let Σ be a σ-algebra of its subsets which contains all singletons.

Definition 10.1. Suppose that we have a countably additive MC on (X, Σ) with operators T and A and transition function $p(x, E)$. Define two operators $T_\gamma \overset{\text{def}}{=} rTr^{-1}$ and $A_\gamma \overset{\text{def}}{=} [r^*]^{-1}Ar^*$, which, by construction, act as follows: $T_\gamma\colon C(\gamma X) \to C(\gamma X)$ and $A_\gamma\colon \mathit{rca}(X, \mathcal{B}_\gamma) \to \mathit{rca}(X, \mathcal{B}_\gamma)$, where $\mathcal{B}_\gamma = \mathcal{B}_{\gamma X}$. We call T_γ and A_γ the γ-*operators* (or the *operators* of the γ-MC, which is not defined yet).

The following properties of T_γ and A_γ are immediate from the definition and the fact that r is an isometry. For convenient references, we gather them:

Theorem 10.1. *For every countably additive* MC, *the operators T_γ and A_γ are linear, bounded, positive, A_γ is isometric on the cone of positive elements, $\|T_\gamma\| = \|A_\gamma\| = 1$, and $T_\gamma^* = A_\gamma$.*

Thus, T_γ and A_γ have all the basic properties of Markov operators. It remains to find their integral representations.

Theorem 10.2. *For every countably additive* MC, *there exists a function $q\colon \gamma X \times \mathcal{B}_\gamma \to [0, 1]$ such that*

$$\begin{aligned} &q(x, \cdot) \in \mathit{rca}(\gamma X, \mathcal{B}_\gamma), && x \in \gamma X;\\ &q(\cdot, E) \in B(\gamma X, \mathcal{B}_\gamma), && E \in \mathcal{B}_Z(\gamma X);\\ &q(\cdot, E) \in C(\gamma X), && E \in \mathcal{N}_{\gamma X};\\ &q(x, \gamma X) = 1, && x \in \gamma X, \end{aligned}$$

and T_γ and A_γ are integrally representable via the kernel $q(x, E)$:

$$(T_\gamma f)(x) = T_\gamma f(x) = \int_{\gamma X} f(y) q(x, dy), \qquad f \in C(\gamma X),\ x \in \gamma X;$$

$$(A_\gamma \mu)(E) = A_\gamma \mu(E) = \int_{\gamma X} q(x, E) \mu(dx), \quad \mu \in \mathit{rca}(\gamma X, \mathcal{B}_\gamma),\ E \in \mathcal{B}_\gamma.$$

Proof. By Bartle and Dunford's theorem (see [9, Chapter VI, Item 7, Theorem 1]) on the general form of a bounded linear operator in the space of continuous functions on a Hausdorff compact set, there exists a mapping $\widetilde{q}$ such that

$$\widetilde{q}\colon \gamma X \to C^*(\gamma X) = rca(\gamma X, \mathcal{B}_\gamma);$$
$$T_\gamma f(x) = \langle f, \widetilde{q}(x)\rangle, \quad f \in C(\gamma X),\ x \in \gamma X;$$
$$\|T_\gamma\| = \sup_{x\in\gamma X} \|\widetilde{q}(x)\|;$$

moreover, $\widetilde{q}(\cdot)$ is continuous in the topology τ_C of $rca(\gamma X, \mathcal{B}_\gamma)$.

So, at every $x \in \gamma X$, $\widetilde{q}(x)$ is a regular countably additive measure on $\mathcal{B}_\gamma$, which we denote by $q(x, \cdot) \in rca(\gamma X, \mathcal{B}_\gamma)$. We have

$$T_\gamma f(x) = \langle f, \widetilde{q}(x)\rangle = \int_{\gamma X} f(y)q(x, dy).$$

Assume that $f = \chi_E$, where χ_E is the characteristic function of a set $E \in \mathcal{N}_{\gamma X}$. Clearly, $\chi_E \in C(\gamma X)$ and χ_E is in the domain of T_γ. Since T_γ is positive, it follows that $T_\gamma \chi_E(x) = q(x, E) \geq 0$ for all $x \in \gamma X$ and $E \in \mathcal{N}_{\gamma X}$.

Suppose that $q(x, E) = -\alpha < 0$ for some $x \in \gamma X$ and $E \in \mathcal{B}_\gamma$. By regularity of the measure $\eta(\cdot) = q(x, \cdot)$, for every $\varepsilon > 0$, there exist $F = \overline{F} \subset E$ and $G = \mathring{G} \supset E$ such that $\mathrm{Var}(\eta, G \setminus E) < \varepsilon$, $\mathrm{Var}(\eta, E \setminus F) < \varepsilon$, and $\mathrm{Var}(\eta, G\setminus F) < 2\varepsilon$. From this it is easy to see that $-\alpha-\varepsilon \leq \eta(G) \leq -\alpha+\varepsilon$ and $-\alpha - \varepsilon \leq \eta(F) \leq -\alpha + \varepsilon$. The algebra $\mathcal{N}_{\gamma X}$ is a base of topology in γX, and the set $F = \overline{F} \subset \gamma X$ is compact. Consequently, there exists $U \in \mathcal{N}_{\gamma X}$ such that $F \subset U \subset G$. As was proven above, $U \in \mathcal{N}_{\gamma X}$ satisfies $\eta(U) = q(x, U) \geq 0$. Then it is easy to check that $\mathrm{Var}(\eta, U \setminus F) \geq \alpha - \varepsilon$ and $\mathrm{Var}(\eta, G \setminus U) \geq \alpha - \varepsilon$, whence $\mathrm{Var}(\eta, G \setminus F) \geq 2\alpha - 2\varepsilon$. Taking, for instance, $\varepsilon = \frac{\alpha}{4}$, we arrive at a contradiction to the inequality $\mathrm{Var}(\eta, G \setminus F) < 2\varepsilon$. Therefore, $q(x, E) \geq 0$ for all $x \in \gamma X$ and $E \in \mathcal{B}_\gamma$.

Since r sends the function $\chi_X \in B(X, \Sigma)$ to the function $\chi_{\gamma X} \in C(\gamma X)$, by definition and the properties of T_γ, we obtain $T_\gamma \chi_{\gamma X} = \chi_{\gamma X}$, whence $q(x, \gamma X) \equiv 1$.

Suppose that $E \in \mathcal{N}_{\gamma X}$, i.e., that E is clopen. Then $\chi_E \in C(\gamma X)$, and, by construction, $T\chi_E(\cdot) = q(\cdot, E) \in C(\gamma X)$, i.e., $q(\cdot, E)$ is a continuous function.

Since the algebra $\mathcal{N}_{\gamma X}$ generates the σ-algebra $\mathcal{B}_Z(\gamma X)$ of Baire subsets in γX, a result by Foguel (see [13]) implies that, for every Baire set $E \in \mathcal{B}_Z(\gamma X)$, $q(\cdot, E)$ is a Baire function as well, i.e. $q(\cdot, E) \in B\big(\gamma X, \mathcal{B}_Z(\gamma X)\big)$. All Baire functions are Borel; hence $q(\cdot, E) \in B(\gamma X, \mathcal{B}_\gamma)$.

By another Bartle and Dunford's theorem (see [9, Chapter VI, Item 7, Theorem 2]) on the general type of integral operators, the adjoint operator A_γ to T_γ also has an integral representation by means of the kernel $q(x, E)$:

$$A_\gamma \mu(E) = \int_{\gamma X} q(x, E)\mu(dx) \quad \text{for all} \quad \mu \in rca(\gamma X, \mathcal{B}_\gamma) \quad \text{and} \quad E \in \mathcal{B}_\gamma.$$

The theorem is proven.

So, the integral kernel $q(x, E)$ in Theorem 10.2 satisfies all basic properties of a transition function (probability) of a Markov chain, and T_γ and A_γ are Markov operators of the chain. Clearly, the kernel $q(x, E)$ is unique for this MC.

Definition 10.2. Given a countably additive MC on (X, Σ), we call the corresponding MC on $(\gamma X, \mathcal{B}_\gamma)$ with transition function $q(x, E)$ and Markov operators T_γ and A_γ of Theorem 10.2 the γ-MC or the *Feller gamma-extension of the* MC *to the gamma-compactification.*

Theorem 10.3. *Each countably additive* MC *on* (X, Σ) *has a unique Feller countably additive extension to* $(\gamma X, \mathcal{B}_\gamma)$. *Moreover, there is a one-to-one correspondence between countably additive* MC*'s on* (X, Σ) *and a class of countably additive Feller* MC*'s on* $(\gamma X, \mathcal{B}_\gamma)$. *Treating* MC*'s as Markov operators, this correspondence is an isometric isomorphism between the classes of operators.*

The claim of Theorem 10.3 is nothing but a reformulation of the above properties of $T = r^{-1}T_\gamma r$, $A = r^* A_\gamma [r^*]^{-1}$, r, and $[r^*]^{-1}$.

Denote by $\mathcal{P}_{ca}$ the class of all countably additive MC's on the initial phase space (X, Σ).

We now look at our construction from a different viewpoint. Let $(\gamma_\Sigma X, \tau_\gamma)$ be the gamma-compactification of the initial measure (generally speaking, nontopological) space (X, Σ). The compact space $(\gamma_\Sigma X, \tau_\gamma)$ can be considered independently of whether we have an MC on (X, Σ) or not.

On $(\gamma_\Sigma X, \mathcal{B}_\gamma)$, consider the *class* $\gamma\mathcal{P}$ *of all countably additive Feller* MC with countably additive transition functions $q(x, E)$ and corresponding Markov operators

$$T_\gamma \colon C(\gamma X) \to C(\gamma X), \quad A_\gamma \colon rca(\gamma X, \mathcal{B}_\gamma) \to rca(\gamma X, \mathcal{B}_\gamma).$$

Theorem 10.3 establishes an isomorphism between $\mathcal{P}_{ca}$ and a part of $\gamma\mathcal{P}$. There arises a natural question: What part of $\gamma\mathcal{P}$ corresponds to $\mathcal{P}_{ca}$. Does the whole $\gamma\mathcal{P}$ happen to be isomorphic with $\mathcal{P}_{ca}$? The answer to this is in the negative.

To find this out, it is time to recall the finitely additive MC's on (X, Σ) which have been considered in Section 5. As before, by a finitely additive

MC on (X,Σ), we mean an MC with finitely additive transition function, i.e., with a function $p(x,E)$ satisfying the condition $p(x,\cdot)\in ba(X,\Sigma))$ for all $x\in X$. Denote by $\mathcal{P}_{ba}$ the class of all finitely additive MC's on the initial phase space (X,Σ).

We now make a remark. Obviously, $\mathcal{P}_{ca}\subset\mathcal{P}_{ba}$. Since, for every $x\in X$, the transition function (measure) splits into the sum $p(x,\cdot)=p_1(x,\cdot)+p_2(x,\cdot)$ of its countably additive and purely finitely additive components, we can say that $\mathcal{P}_{ba}$ splits into a sum of the classes $\mathcal{P}_{ca}$ and $\mathcal{P}_{pfa}$. We will not specify what was said in the previous sentence since we are not going to use any specific formalization of such a decomposition (which is not an easy matter).

Suppose that we have a finitely additive MC on (X,Σ) with operators T and A and finitely additive transition function $p(x,E)$. As for a countably additive MC in Definition 10.1, define two γ-operators: $T_\gamma = rTr^{-1}$ and $A_\gamma=[r^*]^{-1}Ar^*$. These operators act as follows:

$$T_\gamma\colon C(\gamma X)\to C(\gamma X),\quad A_\gamma\colon rca(\gamma X,\mathcal{B}_\gamma)\to rca(\gamma X,\mathcal{B}_\gamma).$$

It is easy to see that Theorem 10.1 holds for a finitely additive MC. The operators T_γ and A_γ are linear, bounded, positive, A_γ is an isometry in the cone, $\|T_\gamma\|=\|A_\gamma\|=1$, and $T_\gamma^*=A_\gamma$.

Theorem 10.2 is based on the Bartle–Dunford theorems on the general form of an *arbitrary* bounded linear operator in the space of continuous functions on a compact space. Repeating the proof of Theorem 10.2 literally, we obtain the following assertion.

Theorem 10.4. *For every finitely additive* MC *on* (X,Σ), *there exists a function* $q\colon\gamma X\times\mathcal{B}_\gamma\to[0,1]$ *meeting all assertions of Theorem* 10.2. *Moreover, the operators* T_γ *and* A_γ *are integrally representable via the kernel* $q(x,E)$ *by means of the integral formulas of Theorem* 10.2.

As in Definition 10.2, for a finitely additive MC on (X,Σ), we call the MC on $(\gamma X,\mathcal{B}_\gamma)$ corresponding to it by Theorem 10.4 the γ-MC or the *Feller* γ*-extension of the initial* MC *to* $(\gamma X,\mathcal{B}_\gamma)$.

In the same way, we obtain an analog to Theorem 10.3.

Theorem 10.5. *Each finitely additive* MC *on* (X,Σ) *has a unique Feller countably additive extension to* $(\gamma X,\mathcal{B}_\gamma)$. *In addition, there are a one-to-one correspondence between all finitely additive* MC*'s and a class of all countably additive Feller* MC*'s on* $(\gamma X,\mathcal{B}_\gamma)$ *and an isometric isomorphism between the corresponding classes of operators.*

Remark. We have not combined the pairs of theorems one of which is a generalization of the other deliberately, respectful to the psychologically deeply rooted tradition to consider the countably additive probability theory separately from the finitely additive version. Now it transpires that there is no harm in such unification, at least, for the functional theory of Markov chains.

Thus, there are enough different Feller MC's on $(\gamma_\Sigma X, \mathcal{B}_\gamma)$ in $\gamma\mathcal{P}$ for a one-to-one correspondence with all finitely additive MC's in $\mathcal{P}_{ba}$. We are left with proving that this exhausts $\gamma\mathcal{P}$.

Theorem 10.6. *Each countably additive Feller* MC *in* $\gamma\mathcal{P}$ *defined on* $(\gamma_\Sigma X, \mathcal{B}_\gamma)$ *is the Feller* γ*-extension of a finitely additive* MC *in* $\mathcal{P}_{ba}$ *defined on* (X, Σ).

Proof. Let a Feller MC on $(\gamma_\Sigma X, \mathcal{B}_\gamma)$ have transition function $q(x, E)$ and Markov operators T_γ and A_γ. Put $T = r^{-1}T_\gamma r$ and $A = r^* A_\gamma [r^*]^{-1}$.

It suffices to prove that there exists a function $p\colon X \times \Sigma \to [0, 1]$ for which

$$
\begin{aligned}
&p(x, \cdot) \in ba(X, \Sigma), && x \in X;\\
&p(\cdot, E) \in B(X, \Sigma), && E \in \Sigma;\\
&p(x, X) = 1, && x \in X,
\end{aligned}
$$

the operators T and A are such that

$$
T\colon B(X, \Sigma) \to B(X, \Sigma), \quad A\colon ba(X, \Sigma) \to ba(X, \Sigma), \quad T^* = A
$$

and are integrally representable via the kernel $p(x, E)$:

$$
\begin{aligned}
(Tf)(x) = Tf(x) &= \int_X f(y)p(x, dy), && f \in B(X, \Sigma),\ x \in X;\\
(A\mu)(E) = A\mu(E) &= \int_X p(x, E)\mu(dx), && \mu \in ba(X, \Sigma),\ E \in \Sigma.
\end{aligned}
$$

We now prove this. Clearly, by construction, T and A act in the above-indicated spaces. Moreover, they are positive, and A is an isometry on the cone. For every $E \in \Sigma$, we have $T\chi_E \in B(X, \Sigma)$. Put $p(x, E) \overset{\text{def}}{=} T\chi_E(x)$. Then $p(\cdot, E) \in B(X, \Sigma)$ and $p(x, E) \geq 0$ for all $x \in X$ and $E \in \Sigma$. By the definition of adjoint operator, every $\mu \in ba(X, \Sigma)$ and $f \in B(X, \Sigma)$ satisfy the equality $\langle f, A\mu\rangle = \langle \mu, Tf\rangle$, i.e., $\int f(x)(A\mu)(dx) = \int (Tf)(x)\mu(dx)$. Hence, putting $f = \chi_E$, we infer that

$$
(A\mu)(E) = \int \chi_E(x)(A\mu)(dx) = \int (T\chi_E)(x)\mu(dx) = \int p(x, E)\mu(dx).
$$

Thus we have an integral representation for A.

Let $\mu = \delta_z$ be the Dirac measure at $z \in X$. The equalities $\delta_z(X) = \|\delta_z\| = 1$ and the fact that A is an isometry imply that $1 = (A\delta_z)(X) = \int p(x, X)\delta_z(dx) = p(z, X)$ for all $z \in X$. Moreover, since $\delta_z \in ca(X, \Sigma) \subset ba(X, \Sigma)$ and $A\colon ba(X, \Sigma) \to ba(X, \Sigma)$, it follows that $A\delta_z = \int p(x, \cdot)\delta_z(dx) = p(x, \cdot) \in ba(X, \Sigma)$, i.e., $p(x, \cdot)$ is finitely additive with respect to the second argument. Hence, in particular, for all $E \in \Sigma$ and $x \in X$, we obtain $0 \leq p(x,E) \leq p(x,X) = 1$.

Now, we can integrate with respect to the measure $p(x,\cdot)$ for an arbitrary $x \in X$. Suppose that $f \in B(X,\Sigma)$ and $\mu = \delta_z$. Then $Tf(z) = \int (Tf)(x)\delta_z(dx) = \int f(x)(A\delta_z)(dx) = \int f(x)p(z,dx)$. The theorem is proven.

Obviously, Theorem 10.6 determines a finitely additive MC on (X,Σ) with finitely additive transition function $p(x,E)$ and the corresponding operators T and A. The construction of T and A by means of the isometric isomorphisms r and r^* guarantees uniqueness of their definition on using the operators of the class $\gamma\mathcal{P}$ of Feller countably additive MC's on $(\gamma_\Sigma X, \mathcal{B}_\gamma)$. Thus, Theorem 10.6 is "converse" to Theorem 10.5. Combining the two assertions, we arrive at the following final statement.

Theorem 10.7. *Let (X,Σ) be a measure space. There is a one-to-one correspondence between all finitely additive* MC*'s in the class $\mathcal{P}_{ba}$ on (X,Σ) and all countably additive Feller* MC*'s in the class $\gamma\mathcal{P}$ on $(\gamma_\Sigma X, \mathcal{B}_\gamma)$. Moreover, there exists an isometric algebraic isomorphism between all Markov operators of finitely additive* MC*'s on (X,Σ) and all Markov operators of countably additive Feller* MC*'s on $(\gamma_\Sigma X, \mathcal{B}_\gamma)$.*

This assertion determines a proper place of the finitely additive Markov chains which took their origin in game theory, and, hopefully, will end disputes on what purpose the finitely additive Markov chains are evoked for.

It is easy to find explicit relations between the transition functions of the initial MC and γ-MC isomorphic to it. The results and construction of Sections 9, 10 immediately imply the following

Theorem 10.8. *The transition functions $p(x,E)$ of the initial finitely additive* MC *on (X,Σ) and $q(z,G)$ of the countably additive Feller γ-*MC *on $(\gamma_\Sigma X, \mathcal{B}_\gamma)$ are related as follows*:

$$
\begin{array}{lll}
(1) & p(x,E) = q\big(s(x), t(E)\big), & x \in X,\ E \in \Sigma; \\
(2) & q(z,G) = p\big(s^{-1}(z), t^{-1}(G)\big), & z \in s(X),\ G \in \mathcal{N}_{\gamma X}; \\
(3) & p(x,\cdot) = r^*\big[q\big(s(x),\cdot\big)\big], & x \in X; \\
(4) & p(\cdot,E) = r^{-1}\big[q\big(\cdot, t(E)\big)\big], & E \in \Sigma; \\
(5) & q(z,G) = r\big[p\big(\cdot, t^{-1}(G)\big)\big](z), & z \in \gamma X,\ G \in \mathcal{N}_{\gamma X}; \\
(6) & q(z,G) = r^{*-1}\big[p\big(s^{-1}(z),\cdot\big)\big](G), & z \in s(X),\ G \in \mathcal{B}_{\gamma X}.
\end{array}
$$

If the initial MC is countably additive then the formulas of the theorem hold too.

Using the above techniques, in the second part of this article we will prove ergodic theorems for Markov chains in which we connect asymptotic behavior of the chains with the properties of invariant finitely additive measures.

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